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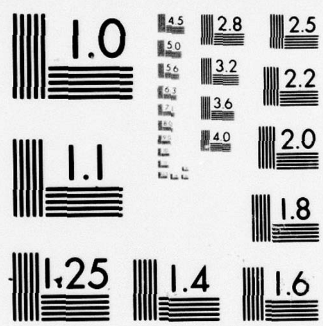

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REPORT SA-78

GENERALIZED OPTIMALITY CRITERIA FOR FREQUENCY CONSTRAINTS, BUCKLING CONSTRAINTS AND BENDING ELEMENTS

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CONTENTS

1. INTRODUCTION

In Ref. [1] new methods for structural weight minimization were presented, which were restricted to finite element models made up of bars and membranes. The only constraints imposed in the optimization problem were concerned with static stress and displacement limitations. The main goal of the present work is to extend the mixed method and the generalized optimality criterion set forth in Ref. [1] to deal with more complex problems involving frequency and buckling constraints as well as bending elements.

It has been shown in Ref. [1] that optimality criteria and mathematical programming approaches to structural optimization have now reached a stage where they use the same basic concepts. Both approaches convert the primary minimization problem into a sequence of explicit approximate problems. To this end conventional optimality criteria techniques are based upon physical understanding of the phenomena involved when redesign takes place (load pathes, stress redistribution, etc...). Another intuitive guideline is employed for deriving recursive redesign formulas. However these optimality criteria techniques can be justified from a mathematical programming point of view, yielding therefore a unified approach to structural optimization. This unified approach will be reviewed in chapter 2 with emphasis on the primal or dual character of the solution scheme used to solve each explicit problem.

As in Ref. [1] the design variables will be taken as the transverse sizes of the structural members, namely, the cross-sectional areas of bar and beam elements and the thicknesses of membrane, plate and flat shell elements. The objective function to be minimized is defined as a linear function of the member sizes a_i :

$$W = \sum_{i=1}^{n} \rho_i \, \ell_i \, a_i \qquad (1.1)$$

where ℓ_i is a geometrical factor such that the product $\ell_i\ a_i$ is the volume of the i^{th} element (length of a one-dimensional element; area of a two-dimensional element). ρ_i denotes a scalar quantity associated with the i^{th} element. In most cases, it is simply the specific weight, in which case the objective function posed by Eq. (1.1) represents the weight of the structure.

The structural optimization problem considered in this work consists in minimizing the objective function embodied in Eq. (1.1) subject to a wide variety of constraints. The behavior constraints impose limitations on the quantities describing the structural response : static stresses and displacements under multiple loading conditions, natural frequencies and linear buckling load factors. In addition, side constraints are directly prescribed on the design variables themselves.

1.1 Side constraints

For various reasons it is generally necessary to impose lower and upper limits on the design variables, which constitutes the so-called side constraints :

$$\underline{a}_i \leq a_i \leq \bar{a}_i \tag{1.2}$$

Note that throughout this work, a lower bar denotes a minimum allowable limit, while an upper bar represents a maximum allowable limit. Although the side constraints embodied in Eq. (1.2) do not correspond really to failure modes of the structure, they are nevertheless important in the problem statement, because they reflect fabrication and analysis validity considerations.

It is also desirable to impose several structural members to be described by the same design variable. Therefore the finite elements are subdivided into some preselected groups, so that one independent design variable is sufficient to control the size of all finite elements pertaining to the same linking group. Design variable linking can be introduced in the form of equality cons-

traints on the member sizes :

$$a = G\,a' \tag{1.3}$$

where a (dimension n) is the full vector of the member sizes, a' (dimension n') is the reduced vector of independent design variables and G (dimension n x n') is a boolean matrix (0 or 1 elements). Design variable linking reduces the number of design variables ($n' < n$) and it facilitates imposition of constraints based on symmetry, fabrication requirements and cost considerations regarding the number of parts to be assembled. In chapter 2 through 6 no attention will be paid to the treatment of the linking constraints expressed in Eq. (1.3), however chapter 7 will indicate how to handle them in a simple way.

1.2. Stress and displacement constraints

A significant class of structural optimization problems is concerned with constraints on the static structural response under a set of loading conditions. Strength constraints are usually taken into account by prescribing allowable limits on the stresses in the various members :

$$\sigma_{k\ell} \leq \bar{\sigma}_k \qquad \begin{array}{l} k = 1,n \\ \ell = 1,c \end{array} \tag{1.4}$$

where $\sigma_{k\ell}$ denotes an adequate representation of the stress state in the k^{th} element under the ℓ^{th} loading condition and $\bar{\sigma}_k$ is the allowable stress limit in this particular element. For stress members independent tension and compression allowables can be specified. In shear panel and isotropic membrane elements, where multiaxial stress states exist, strength constraints are introduced by limiting the value of an equivalent stress, using for example the well known von Mises criterion. In the orthotropic membrane elements employed to model fiber composite lamina, several strength failure criteria can be adopted, for example, the Tsai-Hill criterion.

The displacement constraints considered in Ref. [1] are defined as upper bounds on some linear combinations of the displacement degrees of freedom q used in the finite element model :

$$u_j = b_j^T q \tag{1.5}$$

where b_j is a vector of constants. The linear combination of displacements u_j is called a flexibility. The displacement or flexibility constraints read thus as follows :

$$u_{j\ell} \leq \bar{u}_j \qquad j = 1,f \quad \ell = 1,c \tag{1.6}$$

where $u_{j\ell}$ denotes the value of u_j under the ℓ^{th} loading condition and $\bar{u}_j$ is the upper limit. Note that this treatment of displacement constraints includes the usual nodal displacement constraints, the relative displacement constraints, the slope constraints, etc...

In the displacement finite element method employed in Ref. [1], as well as in the present work, the structural analysis needed for evaluating the stress and flexibility constraints embodied in Eqs. (1.4, 1.6) is performed by solving the systems of linear equations:

$$K\, q_\ell = g_\ell \tag{1.7}$$

where K is the structural stiffness matrix and g_ℓ represents the forces acting in the ℓ^{th} loading condition. Once the displacement vector q_ℓ is known, the stresses in the various members are evaluated from the equations :

$$\sigma_{k\ell} = T_k\, q_\ell \tag{1.8}$$

where T_k is the stress matrix for the k^{th} element. Note that in a membrane element the stress matrix is made up of three rows, while in a bar element, it reduces to one row. The class of finite element models studied in Ref. [1] was restricted to thin-walled structures (assembing of bars and membranes) for which the

stress matrices are independent of the design variables and the stiffness matrix exhibits a linear form in the design variables :

$$K = \sum_{i=1}^{n} a_i \bar{K}_i \qquad (1.9)$$

where $\bar{K}_i$ represents the stiffness matrix of the i^{th} element when $a_i = 1$.

Chapter 3 will be devoted to recalling the main results obtained in Ref. [1]. An important result is that there exists two distinct but equivalent procedures for generating first order explicit approximations of the stress and displacement constraints. Both procedures require that a certain number of additional loading cases be treated in the structural analysis phase. The generalized optimality criterion set forth in Ref. [1] employs first order approximation for all the stress and displacement constraints. It can be written in terms of virtual strain energy densities in the structural members. In conventional optimality criteria techniques, only the displacement constraints are replaced with first order approximations, while the stress constraints are transformed into minimum size limits, which corresponds to adopting zero order explicit approximations (fully stressed design concept). Therefore hybrid optimality criteria can be defined as those based on mixed order approximations of the stress constraints. Note that the results stated in Ref. [1] were concerned with applied loads that are independent of the design variables. They will be extended in this work to the case where the load vectors g_ℓ appearing in Eq. (1.7) depend linearly on the design variables (thermal effects, body forces, etc...).

1.3. Frequency constraints

The natural frequencies of a structure are often constrained to reside within some given critical intervals in order to insure adequate control of integrated mechanical systems, or to prevent resonance phenomena, etc... The frequency constraints will therefore be expressed as follows :

$$\underline{\omega}_j < \omega_j < \bar{\omega}_j \qquad j = 1,m \tag{1.10}$$

where the ω_j's represent the eigenfrequencies ordered in an increasing sequence. Most often only the fundamental natural frequency ω_1 is required to be larger than an allowable limit, in which case Eq. (1.10) reduces to a single behavior constraint. Note however that when the optimization process progresses, a change might occur in the lowest vibration mode, so that it is more reliable to always impose multiple frequency constraints, at least on the two first eigenmodes.

Instead of solving systems of linear equations, structural analysis now consists of solving an eigenproblem

$$K\, q_j - \omega_j^2\, M\, q_j = 0 \tag{1.11}$$

where ω_j is the j^{th} frequency, q_j denotes the corresponding modal displacement vector and M represents the structural mass matrix. It will be seen in chapter 4 that the frequency constraints can be treated exactly like the stress and displacement constraints, by using first order Taylor series expansion with respect to the reciprocal design variables. The resulting explicit problem can still be solved by recoursing to primal or dual algorithms, which corresponds to applying the concept of mixed method or that of generalized optimality criterion, respectively. In particular the optimality criterion can be stated in terms of elastic and kinetic energy densities in the structural members.

1.4 Buckling constraints

In many design projects, stability considerations play an important role in checking off the possible failure modes of the structure. For numerical optimization purposes it is convenient to distinguish two different types of instability phenomena : local instability of an individual structural component and global instability of the entire structure. Local instability

can often be controlled by defining adequate "equivalent" stresses in the strength constraints posed in Eq. (1.4) and/or by reducing the compressive allowable stresses in axial members and panels. For example Euler buckling constraint in a tubular member with specified mean radius r, Young's modulus E and length ℓ is obtained by replacing the compression allowable limit with

$$\bar{\sigma} = \pi^2 \frac{Er^2}{2\ell^2} \qquad (1.12)$$

On the other hand global instability constraints depend in a complex way on all structural components. In the context of a linear analysis, global instability is taken into consideration through an eigenproblem similar to that defining the vibration modes of a structure :

$$K q_j - \lambda_j K_G q_j = 0 \qquad (1.13)$$

where K_G is the geometric stiffness matrix. The eigensolutions of this problem yield the buckling mode vectors q_j and the associated buckling load factors λ_j. When several distinct loading conditions are applied to the structure, buckling constraints must be prescribed under any of the loading conditions, which implies solving several eigenproblems at each redesign stage. So the buckling constraints are generally imposed on several critical loads corresponding to either different buckling modes or to different loading conditions :

$$\underline{\lambda}_j \leq \lambda_j \leq \bar{\lambda}_j \qquad j = 1,m \qquad (1.14)$$

Strictly speaking only the lower bound $\underline{\lambda}_j$ is significant for representing global buckling contraints, with usually $\underline{\lambda}_1 = 1$. However it is sometimes interesting to include multiple constraints with upper bounds $\bar{\lambda}_j$ to make possible the separation of buckling modes that are close to each other. This is useful for reducing the structural sensibility to initial imperfections.

The buckling constraints stated in Eq. (1.14) will be con-

sidered in chapter 5, using again the concepts of mixed method and of generalized optimality criterion. The optimal design will be characterized in terms of strain energy densities in the structural members for the critical buckling modes.

The structural optimization problem studied in the present work consists thus in minimizing the objective function defined in Eq. (1.1), subject to the side constraints given in Eqs.(1.2, 1.3) and to the various behavior constraints embodied in Eqs. (1.4, 1.6, 1.10, 1.14). This problem is examined initially by restricting its formulation to structural models made up of bar, membrane and shear panel elements, which are quite adequate for idealizing thin walled structures subject mainly to extension loads (chapters 2 through 5). Subsequently chapter 6 will be concerned with structural models that are capable of carrying flexural forces, such as beam, plate and shell elements. It is shown that high quality explicit approximation of the behavior constraints can still be generated using first order Taylor series expansion, provided that adequate intermediate variables are selected. The idea of generalized optimality criterion remains fully valid and it keeps its interpretation in terms of energy densities in the structural members.

In chapter 7 some indications will be given about the computer implementation of the optimization strategies proposed in Ref. [1] and continued in the present work. Finally some illustrative examples will be presented in chapter 8. Based on the numerical results obtained, it can be concluded that the concepts developed in this work are rather general. They lead to efficient structural optimization methods, which are capable of dealing with a wide variety of behavior constraints and of structural models. The approach presented converts the primary problem into a sequence of simple explicit problems. The explicit problem statement is obtained by linearizing the behavior constraints with

respect to appropriate intermediate variables. Therefore the method requires numerical evaluation of the constraint gradients at each redesign stage. Using a dual solution scheme for solving each explicit problem leads to generalization of the conventional optimality criteria techniques, with comparable efficiency but increased reliability. Solving partially each explicit problem, by employing a primal solution scheme, yields the mixed method, which facilitates control over convergence of the overall optimization process.

2. MIXED METHOD AND GENERALIZED OPTIMALITY CRITERION

This chapter summarizes some theoritical results obtained in a previous work [1,2]. It is shown that a powerful approach to structural optimization has now emerged, which consists in replacing the primary problem with a sequence of simple explicit problems. In the optimality criteria and mathematical programming approaches, the behavior constraints are approximated using respectively virtual load considerations and linearization with respect to the reciprocal design variables.

An attractive strategy is to solve partially each explicit problem, using a primal solution scheme, before reanalyzing the structure and updating the approximate problem statement. This process facilitates generation of a sequence of steadily improved feasible designs. This approach can be interpreted as a mixed primal-linearization method and it introduces an interesting possibility of controlling the convergence of the optimization process.

An alternative approach is to recognize that the explicit but approximate problem statement is of such high quality that it can be solved exactly, rather than partially. Such an exact solution can be efficiently generated by using a dual solution scheme. This approach is closely related to pure linearization methods in mathematical programming, but it can also be viewed as a rigorous generalization of the conventional optimality criteria techniques.

2.1. Fundamental concepts

For presenting the fundamental concepts used in the methods formulation, the optimization problem stated in chapter 1 is written in the mathematical form :

$$\text{minimize} \qquad W = \sum_{i=1}^{n} \rho_i \ell_i a_i \tag{2.1}$$

subject to

$$h_j(a) \geq 0 \qquad j = 1,m \tag{2.2}$$

$$\bar{a}_i \geq a_i \geq \underline{a}_i \qquad i = 1,n \tag{2.3}$$

where a_i denote the n design variables which correspond to member sizes of either individual finite elements or, more often of some preselected groups of finite elements (see section 1.1). The objective function to be minimized is the structural weight. It is a linear function of the design variables a_i. All the behavior constraints discussed in chapter 1 (see Eqs.1.6,1.14) are gathered in the inequalities expressed in Eq. (2.2). They represent limitations on quantities describing the structural response, for example, the stresses and the displacements under multiple static loading cases, the natural frequencies, the buckling loads, etc... The design variables are also subjected to side constraints that prevent them from becoming too small or too large (Eq. 2.3). Note that the design variables are assumed to vary continuously in Eqs. (2.1-2.3). However discrete variables could also be introduced in the problem statement.

The structural optimization problem embodied in Eqs. (2.1-2.3) is a nonlinear mathematical programming problem to which standard minimization techniques can be applied. However this problem exhibits some characteristics that make it complicated when practical structural design applications are considered. The main difficulty arises from the fact that the behavior constraints (2.2) are in general implicit functions of the design variables and their precise numerical evaluation for a particular design requires a complete finite element analysis. Since the solution scheme is essentially iterative, it involves a large number of structural reanalyses. Therefore the computational cost often becomes prohibitive when large structural systems are dealt with.

The well known necessary KUHN-TUCKER conditions characterize any local minimum of the nonlinear programming problem (2.1-2.3) [3]. They are written as follows for each design variable :

$$\frac{\partial W}{\partial a_i} - \sum_{j=1}^{n} r_j \frac{\partial h_j}{\partial a_i} - v_i + t_i = 0 \qquad i = 1,n \qquad (2.4)$$

with the complementary conditions :

$$r_j h_j = 0 \qquad r_j \geqslant 0 \qquad j = 1,m \qquad (2.5)$$

$$v_i(a_i - \underline{a}_i) = 0 \qquad v_i \geqslant 0 \qquad i = 1,n \qquad (2.6)$$

$$t_i(\bar{a}_i - a_i) = 0 \qquad t_i \geqslant 0 \qquad i = 1,n \qquad (2.7)$$

The quantities (r_j, j = 1,m), associated with the behavioral constraints (2.2), and (v_i, t_i, i = 1,n), associated with the side constraints (2.3), are called dual variables as opposed to the primal variables, which are the a_i. They have the meaning of lagrangian multipliers conjugated to the constraints. Depending upon whether a given constraint becomes an equality or not at the optimum (i.e. is active or inactive), the corresponding dual variable is positive or equal to zero (see Eqs.

2.5-2.7). The KUHN-TUCKER conditions are in general only necessary. In the special case of a convex problem, they become also sufficient and characterize a global optimum. They can then be used to relate the primal variables - i.e. design variables a_i - to the dual variables - i.e. lagrangian multipliers r_j, v_i, t_i -

In the last decade essentially two main approaches have been used to solve the problem (2.1-2.3). One is based on the many rigorous numerical methods of nonlinear mathematical programming. The other uses the more intuitive concepts of optimality criteria. These approaches have often been opposed in the past and two corresponding schools developed. The advantages claimed for the mathematical programming methods are their sound foundations, their convergence properties which can most often be guarantied and their generality which permits consideration of any type of constraints. Their essential disadvantage lies in the computing time which increases rapidly with the size of the problem, leading to unacceptable cost even for relatively simple problems.

The optimality criteria are based on explicit approximations of the behavior constraints which are exact in special cases, most of the time in statically determinate cases. These approximations are supposed, on intuitive basis, to hold in the general case. Then the statement of the problem (2.1-2.3) often disappears since redesign formulae can be derived by various means, like, for instance, the KUHN-TUCKER conditions (2.4-2.7) in which the explicit approximations are introduced. The advantages of using the optimality criteria are the computing cost which is low and the intuitive basis, which is often appealing to the engineer. In fact the number of reanalysis cycles does not increase with the complexity of the structure, nor with the number of design variables. As a consequence of this property, until recently, large scale examples of structural optimization have only been solved by optimality criteria. The essential

disadvantages of the optimality criteria approach are its lack of generality and of sound mathematical foundations, which explains the often impredictable convergence properties and even the convergence to non optimal solutions.

It has been shown in a recent work [1, 2, 4] that these two approaches are in fact related to each other and that the optimality criteria can be viewed as special mathematical programming methods using linearization for the constraint surfaces. The main results of the work reported in Ref. [1] will be summarized in the next three sections of this chapter.

2.2. The explicit approximate problem

The non linear programming problem to be solved involves minimization of an explicit objective function subject to explicit side constraints and implicit behavior constraints. As previously mentioned the implicit character of the functions h_j appearing in Eq. (2.2) constitutes the main difficulty in achieving efficiency in structural optimization methods. That is why recoursing to explicit approximation of the behavior constraints appears to be the only possible way of successfully attacking the problem under consideration.

Most of the optimality criteria approaches deal with problems involving constraints on static stresses and displacements, in which case the behavior constraints of Eq. (2.2) can be written

$$h_j(a) \equiv \bar{u}_j - u_j(a) \geq 0 \tag{2.8}$$

where $\bar{u}_j$ denotes an upper bound to a response quantity $u_j(a)$ (stress, nodal displacement, relative displacement, etc...). Using virtual load considerations, explicit approximations of the behavior constraints (2.8) can be generated :

$$\tilde{h}_j(a) \equiv \bar{u}_j - \sum_{i=1}^{n} \frac{c_{ij}}{a_i} \geq 0 \tag{2.9}$$

where the coefficients c_{ij} are related to virtual energy densities in the structural members.

The way theses coefficients are computed is explained in detail in Ref. [1] and will be briefly reviewed in chapter 3. For the moment it is sufficient to recall that the c_{ij}'s are constant coefficients in the case of a statically determinate structure, so that Eq. (2.9) represents then the exact explicit form of the behavior constraints. In the case of a statically indeterminate structure, the c_{ij}'s depend implicitly on the design variables because structural redundancy leads to redistribution of the internal forces when the member sizes are modified. Therefore the explicit constraints embodied in Eq. (2.9) constitute in general approximate forms of the original constraints (2.8). As shown in Ref. [1] the basic idea in the optimality criteria approach can be viewed as transforming the initial implicit problem into a sequence of explicit problems. Each explicit problem results from replacing the behavior constraints (2.8) by their approximate expressions(2.9).

On the other hand the mathematical programming approach, after a period of unefficiency, has finally evolved into a

powerful and now well established design procedure that is also based upon explicit approximation of the behavior constraints. Such an approach was developed independently by SCHMIT [5,6,7,8] as the "approximation concepts approach" and by FLEURY [1,9,10,11] as the "mixed method". Both methods proceed by constructing a high quality explicit approximation of the initial problem and solving it partially, using a primal mathematical programming algorithm, before reanalyzing the structure and updating the approximate problem statement. This process facilitates generation of a sequence of steadily improved feasible designs, which is an attractive feature for practical design purposes.

The key idea in the mixed method [1,9-11], as well as in the approximation concepts approach [5-8], is to linearize the behavior constraints with respect to the reciprocal design variables

$$x_i = \frac{1}{a_i} \tag{2.10}$$

Justification for this change of variables lies in the fact that the stresses and the displacements are linear functions of the x_i's for a statically determinate structure. Therefore in a moderately hyperstatic case, it can be expected that the constraint surfaces are very shallow and close to planes in the reciprocal design variable space. It is then possible, in a method of projected gradient, to progress with much larger steps along tangent planes without seriously violating the constraints. Moreover the shallowness of the constraint surfaces implies that their linearized forms are usually very good approximations and therefore the structure does not need to be reanalyzed after each iteration in a mathematical programming algorithm. The linearized behavior constraints are obtained using first order Taylor series expansion in terms of the reciprocal variables x_i :

$$\tilde{h}_j(x) = \bar{u}_j - [\, u_j^o + \sum_{i=1}^{n} \left(\frac{\partial u_j}{\partial x_i}\right)^o (x_i - x_i^o) \,] \geq 0 \tag{2.11}$$

where the upperscript ^{o} denotes quantities evaluated at the actual design point x^o, where the structural analysis is performed. Note that the finite element analysis capability must include auxiliary sensitivity analyses for evaluating first partial derivatives of the response quantities. Most often the well known pseudo-loads technique is employed (see section 3.1.2).

Now it can be shown [see Refs. 1 and 4] that the explicit approximations of the behavior constraints used in both the optimality criteria and mathematical programming approaches (Eqs. 2.9 and 2.11, respectively) are identical. Indeed the virtual energy densities c_{ij} employed in the optimality criteria approaches are nothing else that the gradients of the response quantities with respect to the reciprocal variables :

$$c_{ij} \equiv \frac{\partial u_j}{\partial x_i} \qquad (2.12)$$

Furthermore the definition of the c_{ij}'s following from the virtual load technique clearly indicates that

$$u_j^o \equiv \sum_{i=1}^{n} c_{ij}^o \; x_i^o \qquad (2.13)$$

Therefore Eq. (2.11) can be rewritten

$$\tilde{h}_j(x) \equiv \bar{u}_j - \sum_{i=1}^{n} c_{ij}^o \; x_i \geq 0 \qquad (2.14)$$

which is equivalent to Eq. (2.9).

In conclusion a unified structural optimization approach has emerged, which consists in replacing the initial problem (2.1-2.3) with a sequence of explicit approximate - or linearized - problems. Each linearized problem exhibits the following form when written in terms of the reciprocal variables :

minimize $$W = \sum_{i=1}^{n} \frac{\rho_i \ell_i}{x_i} \tag{2.15}$$

subject to

$$\bar{u}_j - \sum_{i=1}^{n} c_{ij}\, x_i \geq 0 \qquad j = 1,m \tag{2.16}$$

$$\underline{x}_i \leq x_i \leq \bar{x}_i \qquad i = 1,n \tag{2.17}$$

where $\underline{x}_i = 1/\bar{a}_i$ and $\bar{x}_i = 1/\underline{a}_i$ are the new side constraints.
It is important to mention that this basic approach of transforming the initial problem into a sequence of explicit problems is now widely recognized and it is routinely employed for large scale industrial applications (see PETIAU [12]).
Various solution schemes are available to treat the explicit problem embodied in Eqs. (2.15-2.17). Depending upon their primal or dual character the convergence properties of the whole optimization process will be different.

2.3. The mixed method

The linearized problem (2.15-2.17) is still a nonlinear programming problem (because the objective function keeps its non linear character), however it is now explicit and easily treated by standard minimization techniques. In order to maintain a primal philosophy (sequence of steadily improved feasible designs), the approximation concepts approach, as initially proposed by SCHMIT and MIURA [6,7], employed either a feasible direction method or an interior penalty function method to solve partially the linearized problem. In this way it is possible to preserve, at each stage of the process, the feasibility of the design point with respect to the primary problem (2.1-2.3). Later the interior penalty function method (called NEWSUMT) only was retained [8,13]. So the NEWSUMT optimizer of the ACCESS-3 program [13,14] tends to generate a sequence of design points that "funnels down the middle" of the

feasible region. This represents an attractive feature from an engineering point of view since it provides the design with acceptable non critical designs at each stage of the process. Furthemore the convergence properties of the method can be controlled by modifying the number of response surfaces (typically 1 or 2) and the response factor decrease ratio (typically 0.5 through 0.1).

On the other hand, starting from an optimality criteria approach, a method similar to the approximation concepts approach was independently developed in Refs. [9-11]. This mixed method uses virtual load considerations to generate a first order approximate problem which is identical to the linearized problem posed by Eqs. (2.15-2.17). This problem is also solved partially using a primal solution scheme, with the aim of preserving the design feasibility as in the approximation concepts approach. Each iteration in the mixed method is made up of a restitution phase and a minimization phase.

The restitution phase consists in bringing the design point back to the boundary of the feasible region. It is based upon the concept of scaling, which is classical in the optimality criteria approaches. Scaling does not introduce any redistribution of the internal forces in the structure and therefore it permits generation of a feasible boundary point without any additional structural analysis. In the design space scaling corresponds to a move along a straight line joining the origin to the current design point, where the structural analysis is made (see Fig. 1). Along the scaling line the coefficients c_{ij} (see Eq. 2.9) remain constant and the explicit approximations lead to the exact values of the constraints and their gradients (see section 3.1.1). As a result the linearized problem statement is not affected by scaling. Geometrically each real restraint surface is approximated, in the space of the reciprocal variables, by its tangent plane at its point of intersection with the scaling line (see Fig. 1).

Starting from a feasible boundary point, the minimization phase in the mixed method consists in reducing the objective function in the subspace defined by the tangent planes to the constraint surfaces. This is achieved by solving the linearized problem (2.15-2.17) using a projection algorithm, but stopping the minimization after a limited number of steps, denoted $\bar{k}$, before reaching the minimum of the linearized problem. After $\bar{k}$ one-dimensional minimizations have been accomplished, the structure is reanalyzed, scaling is performed, the linearized problem is reformed and again solved partially.

That optimization process is illustrated on Fig. 2 in a hypothetical 2-dimensional space. When $\bar{k}$ is limited to one step, the method is a strict application of a gradient projection type of algorithm to the original problem (2.1-2.3). The method then exhibits all the properties of the primal mathematical programming approaches, that is high cost but guarantied convergence. When $\bar{k}$ is not limited, the linearized problem is solved completely before reanalyzing the structure and updating the linearized problem statement. The method then becomes an optimality criteria approach (see next section). The corresponding properties of fast but uncertain convergence are to be expected. Finally, when $\bar{k}$ is limited to a given finite number, the method is mixed. This leads to considering the number of steps $\bar{k}$ as a convergence control parameter that should be assigned high values for economy and reduced when divergence occurs.

As implemented in the SAMCEF program [15,16], the minimization algorithms used in the mixed method are based either on the conjugate gradient method with an orthogonal projection operator or on the generalized Newton's method with a weighed projection operator. This latter method has also been introduced in the ACCESS-3 program [13,14].

From a mathematical programming point of view, the approximation concepts approach and the mixed method can be classified as mixed primal-linearization methods. The initial problem is transformed into a sequence of linearized problems, which is classical in the linearization methods of mathematical programming. However each subproblem is solved only partially using a primal solution scheme that insures feasibility of the intermediate designs at each stage.

2.4. The generalized optimality criterion

In the previous section a primal philosophy has been adopted that leads to partial solution of the linearized problem (2.15-2.17) using an interior penalty function formulation with a small number of response surfaces or a projection algorithm with a small number of one-dimensional minimization. This primal solution scheme produces a sequence of feasible designs with decreasing values of the objective function. An alternative viewpoint is to recognize that the approximation made by linearizing the constraints with respect to the reciprocal design variables is of such high quality that the current explicit problem can be solved exactly, rather than partially, after each structural reanalysis. This idea leads to abandoning the primal philosophy in favor of a pure linearization approach in mathematical programing, but it can also be viewed as adopting an optimality criteria strategy.

2.4.1. Dual solution scheme

Since only the final exact solution of each linearized problem needs to be known at each redesign stage, any minimization algorithm can be chosen to solve it. In order to improve the computational efficiency, it is advisable to select a specialized nonlinear programming algorithm, well suited to the particular

mathematical structure of the explicit problem (2.15-2.17). The objective function is strictly convex and the constraints are linear, so that the problem (15-17) is a convex programming problem. Moreover all the functions defining this problem are explicit and separable. In such a case at least two kinds of method seem to be efficient : the second order primal projection methods and the dual methods[see Ref. 17].

The dual method formulation is especially attractive, because the dual problem presents a much simpler form than the primal problem. Indeed, for a convex problem, the lagrangian multipliers associated with the constraints have the meaning of dual variables in terms of which an auxiliary and equivalent problem can be stated. Under some unrestrictive conditions, this dual problem can be reduced to the maximization of the lagrangian functional with simple nonnegativity requirements on the dual variables. Since, in addition, the explicit problem (2.15-2.17) is of separable form, the dual formulation leads to a very efficient solution scheme since each primal variable can be analytically expressed in terms of the dual variables. Futhermore the dimensionality of the dual problem is equal to the number of linearized behavior constraints (2.16), which is most often small when compared to the number of design variables. Therefore the dual problem exhibits a simpler form and a lower dimensionality than the primal problem.

From a mathematical programming point of view, the dual methods are thus likely to provide the most efficient solution scheme to the linearized problem (2.15-2.17), provided this problem may be solved exactly at each stage, rather than partially, using a primal method [see Ref. 18]. This demands that the original behavior constraints exhibit a moderate nonlinearity with respect to the reciprocal variables, which is actually true for problems involving stress, displacement, frequency and buckling constraints. Another significant advantage of the dual

methods is that they can take into consideration discrete design variables without weakening the efficiency of the optimization process [13,19]. Discrete variables are useful for representing comercially available gage sizes of sheet metal, the number of plies in a laminated composite skin, etc... Finally duality techniques can also be employed to monitor the progress of the optimization process, as proposed by MORRIS et al. [20].

Two maximization algorithms are available in the SAMCEF program to solve the dual problem : a first order conjugate gradient type of algorithm and a second order Newton type of algorithm [1,16]. Recently these dual algorithms were implemented in the ACCESS-3 program, which is based on the approximation concepts approach [13,14]. The second order algorithm was improved by employing a simple but efficient one-dimensional maximization scheme. The first order one was deeply modified and it is now capable of treating problems involving discrete design variables.

2.4.2. Explicit redesign relations

Turning now to an optimality criteria strategy, the simple form of the linearized problem (see Eqs. 2.15-2.17) suggeststo make use of its explicit character in order to derive redesign formulas for the design variables. In fact, as shown in Ref. [4], the whole process of combining the linearization of the behavior constraints with respect to the reciprocal design variables and a dual solution scheme can be viewed as a generalization of the optimality criteria approaches. This process consists in solving exactly, after each structural reanalysis, the linearized problem (2.15-2.17), which can be recast as follows in terms of the direct design variables :

minimize $$W = \sum_{i=1}^{n} \rho_i \, \ell_i \, a_i \qquad (2.18)$$

subject to
$$\bar{u}_j - \sum_{i=1}^{n} \frac{c_{ij}}{a_i} \geq 0 \qquad j = 1,m \qquad (2.19)$$

$$\bar{a}_i \geq a_i \geq \underline{a}_i \qquad i = 1,n \qquad (2.20)$$

Instead of employing primal or dual mathematical programming methods, an alternative approach, which is typical of the optimality criteria philosophy, is to use the explicit character of the approximate problem (2.18-2.20) in order to express analytically the optimal design variables. This can be achieved using the KUHN-TUCKER conditions (2.4-2.7) which, in view of the convexity of the problem, are sufficient for global optimality. These conditions lead to a generalized optimality criterion yielding explicitly the design variables :

<u>active design variables</u> :

$$a_i = \left(\frac{1}{\rho_i \ell_i} \sum_j c_{ij} \, r_j\right)^{1/2} \quad \text{if} \quad \rho_i \ell_i \underline{a}_i^2 < \sum_j c_{ij} \, r_j < \rho_i \ell_i \bar{a}_i^2 \tag{2.21}$$

<u>passive design variables</u> :

$$a_i = \underline{a}_i \qquad \text{if} \qquad \sum_j c_{ij} \, r_j \leqslant \rho_i \ell_i \underline{a}_i^2 \tag{2.22}$$

$$a_i = \bar{a}_i \qquad \text{if} \qquad \sum_j c_{ij} \, r_j \geqslant \rho_i \ell_i \bar{a}_i^2 \tag{2.23}$$

In these expressions, the lagrangian multipliers r_j are associated with the linearized behavior constraints (2.19). They must satisfy the complementarity conditions (2.5), namely :

<u>active constraint</u> :

$$r_j > 0 \qquad \text{if} \qquad \sum_i \frac{c_{ij}}{a_i} = \bar{u}_j \tag{2.24}$$

<u>inactive constraint</u> :

$$r_j = 0 \qquad \text{if} \qquad \sum_i \frac{c_{ij}}{a_i} < \bar{u}_j \tag{2.25}$$

The equations (2.21-2.23) relating the design variables a_i to the lagrangian multipliers r_j provide a basis for separating the design variables in two groups. The passive variables are those that are fixed to a lower or an upper limit (see Eqs. 2.22

and 2.23, respectively) while the active variables are explicitly given in terms of the lagrangian multipliers using Eq. (2.21). This subdivision of the design variables into active and passive groups is classical in the optimality criteria approaches [21-29]. Equations (2.22, 2.23) clearly indicate that the dual space - i.e. the space of the lagrangian multipliers r_j - is partitioned into several regions corresponding to different divisions in active and passive design variables. These regions are separated from each other by planes across which the second derivatives of the lagrangian function are discontinuous (see Refs. [1,30] for more details).

When the lagrangian multipliers satisfying Eqs. (2.24,2.25) are known, the optimal design variables can be easily computed using the explicit optimality criterion (2.21-2.23). Therefore the problem has been replaced with a new one, which is defined in terms of the lagrangian multipliers only. To solve this new problem, the conventional optimality criteria techniques usually make the assumption that the set of active constraints is known in advance, avoiding thus the inequality constraints on the lagrangian multipliers appearing in Eqs. (2.24,2.25). An update procedure for the retained lagrangian multipliers is then employed, so that the optimal design variables can be sought iteratively by coupling the update procedure and the explicit optimality criterion (2.21-2.23). The essential difficulties involved in applying these optimality criteria methods are those associated with identifying the correct set of active constraints and the proper corresponding set of passive members [25,28]. These difficulties were recognized and addressed with varying degrees of success in studies such as those reported in Refs. [26,27]. However it was only with the dual formulation set forth in Refs. [1,30] that these obstacles were conclusively overcome.

The dual method approach consists in maximizing the lagrangian function subject to nonnegativity constraints on the lagrangian

multipliers. This approach can therefore be viewed as using an update procedure for the lagrangian multipliers, exactly like the conventional optimality criteria techniques. After the update procedure is completed, the design variables can be evaluated using the optimality criteria equations (2.21-2.23). Since the dual maximization problem exhibits a remarkably simple form, its exact solution can be generated at a low computational cost, which is comparable to that required by the recursive techniques of conventional optimality criteria approaches. The dual algorithms can handle a large number of inequality constraints and they intrinsically contain a rational scheme for identifying the active constraints through the non-negativity constraints on the lagrangian multipliers (or dual variables). They also automatically sort out the active and passive design variable groups using the explicit relationships between primal and dual variables.

When the design variables, instead of varying continuously, can only take on available discrete values, it can be shown that the optimality criteria equations (2.21-2.23) must read as follows for each discrete variable (see Refs. [13] and [19]) :

$$a_i = a_i^k \quad \text{if} \quad \rho_i \ell_i a_i^k a_i^{k-1} < \sum_j c_{ij} r_j < \rho_i \ell_i a_i^k a_i^{k+1} \tag{2.26}$$

where it is understood that $\{a_i^k,\ k = 1,2,\ldots\}$ denotes the set of available discrete values for the i^{th} design variable. These expressions show that the dual space is subdivided into several regions, each of which corresponds to a distinct combination of available discrete values of the design variables. These regions are separated from each other by planes across which the first derivatives of the dual function are discontinuous (see Ref. [13,19] for more details).

In summary a generalized optimality criteria approach can be defined in the mathematical programming terminology as a special form of the linearization methods. It amounts to re-

placing the original problem with a sequence of linearized problems and applying a dual solution scheme to each sub-problem. It should be noted however that only the behavior constraints are linearized with respect to the reciprocal design variables, while in classical linearization methods, the objective function is linearized too. This is not necessary in the present approach because the structural weight is an exact explicit function of the reciprocal variables. It should be also emphasized that unlike the primal algorithms discussed in section 2.3, the dual algorithms cannot be used to solve only partially the approximate problem embodied in Eqs. (2.18-2.19), because intermediate points in the dual space usually correspond to higly infeasible points in the primal space. Therefore the possibility of controlling the convergence of the optimization process disappears when a dual optimizer is selected in the SAMCEF [15,16] or ACCESS-3 [13,14] programs.

3. STATIC CONSTRAINTS ON STRESSES AND DISPLACEMENTS

This chapter is concerned with structural optimization problems involving behavior constraints on static stresses and displacements. Most of the material presented recalls conclusions drawn in Ref. [1], however, some new developments will also be reported with regards to applied loads that depend on the design variables.

3.1. Evaluation of the constraint gradients

As emphasized in chapter 2, the mixed method and the generalized optimality criterion set forth in Ref. [1] are both based upon linearization of the stress and displacement constraints. Such a linearization process requires gradient evaluations. Two different procedures are available to this end, namely the virtual load and the pseudo-loads techniques.

3.1.1. The virtual load technique

Most of the conventional optimality criteria approaches, as well as the generalized optimality criterion reported in ref. [1], use the virtual load technique to generate explicit approximations of the stress and displacement constraints. In Ref. [1] a flexibility is defined as a linear combination of displacements :

$$u_j = b_j^T q \tag{3.1}$$

where b_j is a vector of constants. Considering a virtual load vector given numerically by b_j, it follows that the j^{th} flexibility under the ℓ^{th} loading case can be expressed as the sum of the contributions of each element :

$$u_{j\ell} = b_j^T q_\ell = q_j^T K q_\ell = \sum_{i=1}^{n} \frac{c_{ij\ell}}{a_i} \tag{3.2}$$

with

$$c_{ij\ell} = (q_j^T \, K_i \, q_\ell) \; a_i \qquad (3.3)$$

In these expressions q_j and q_ℓ are respectively the virtual and real displacement vectorsand K_i denotes the stiffness matrix of the i^{th} finite element (see Eq. 1.9). The flexibility coefficients $c_{ij\ell}$ are constant for a statically determinate structure, in which case Eq. (3.2) furnishes the exact explicit expression of the flexibilities. Therefore explicit optimality criteria can be derived, which define analytically the optimal values of the design variables when displacement constraints are imposed.

In a finite element model the stresses are also linear combinationsof the displacements, just as the flexibilities defined in Eq. (3.1). Indeed a given stress component in the k^{th} element is written as

$$\sigma_k = t_k^T \, q \qquad (3.4)$$

where t_k is a vector of constants corresponding to a line of the stress matrix (see Eq. 1.8). Following out the above procedure the vector t_k is introduced as a virtual loading case, which permits generation of an explicit expression for the corresponding stress component under the ℓ^{th} load condition :

$$\sigma_{k\ell} = t_k^T \, q_\ell = q_k^T \, K \, q_\ell = \sum_{i=1}^{n} \frac{d_{ik\ell}}{a_i} \qquad (3.5)$$

with

$$d_{ik\ell} = (q_k^T \, K_i \, q_\ell) \; a_i \qquad (3.6)$$

In these equations, q_k denotes the virtual displacement vector due to the virtual load case given by t_k.In a plate element the constraint is usually placed on an equivalent stress whose square is a quadratic form of the displacements. It has been shown in Ref. [1] that an explicit expression of the form given by Eq. (3.5) can still be obtained by using special virtual load cases.

So the explicit expressions of the stress and displacement constraints exhibit now the same form (see Eqs. 3.2 and 3.5). Writing then under the common notations stated in Eq. (2.9), it appears that an optimality criteria approach proceeds by replacing the exact behavior constraints

$$u_j \leq \bar{u}_j \tag{3.7}$$

by the explicit constraints

$$\tilde{u}_j = \sum_{i=1}^{n} \frac{c_{ij}}{a_i} \leq \bar{u}_j \tag{3.8}$$

Introducing virtual strain energies associated with each finite element,

$$e_{ij} = \frac{1}{2} q^T K_i \tilde{q}_j = \frac{1}{2} \frac{c_{ij}}{a_i} \tag{3.9}$$

the generalized optimality criterion reviewed in section 2.4 takes the form

$$\sum_{j=1}^{m} r_j \varepsilon_{ij} = c^{st} \tag{3.10}$$

where the ε_{ij}'s have the meaning of virtual energy densities :

$$\varepsilon_{ij} = \frac{e_{ij}}{\rho_i \ell_i a_i} \tag{3.11}$$

In expression (3.9), $\tilde{q}_j$ represents the virtual displacement vector for a virtual load case conjugated to the j^{th} behavior constraint. Note that the real load case index ℓ is omitted for sake of clarity. It should be recalled that the optimality criterion posed by Eq. (3.10) applies only to the active design variables (see Eqs. 2.21-2.23). In the special case where only one behavior constraint is assigned (i.e., m = 1 in Eq. 3.10), the optimality criterion states that in the optimal structure the virtual strain energy density is the same in each element.

The main result reported in Ref. [1] is that the virtual load technique used in the optimality criteria approaches generates first order explicit approximations of the stress and displacement constraints. As previously mentioned the flexibility coefficients c_{ij} are constant in a statically determinate structure and they depend implicitly on the design variables in a statically indeterminate one. However it is essential to note that they are not affected by a scaling of the design, that is by a multiplication of all the a_i variables by a common factor. In the design space such a scaling moves the design point along a line joining the origin to the current point. Therefore the explicit expression stated in Eq. (3.8) yields the exact value of the flexibility u_j all along the scaling line. Geometrically this means that the approximate constraint surface $\tilde{u}_j = \bar{u}_j$ passes through the point of intersection of the corresponding exact constraint surface $u_j = \bar{u}_j$ with the scaling line (see Fig. 3).

In addition, it can be shown that the explicit constraint (3.8) furnishes also the exact derivatives of the flexibility u_j :

$$\frac{\partial \tilde{u}_j}{\partial a_i} = \frac{\partial u_j}{\partial a_i} = - \frac{c_{ij}}{a_i^2} \qquad (3.12)$$

for any design point on the scaling line [see Ref. 30]. Therefore the explicit forms (3.8) represent first order approximations of the flexibility (or stress) constraints at all points along the scaling line. Geometrically the approximation made in an optimality criteria approach can thus be viewed as replacing each exact constraint surface by a tangent surface at its point of intersection with the scaling line (see Fig. 3). From Eq. (3.12) it is now apparent that the gradients of the stress and displacement constraints with respect to the reciprocal variables are given by the coefficients c_{ij} (see Eq. 2.12). Also the explicit approximate constraints (3.8) when written in terms of the reciprocal variables (see Eq. 2.16) can be identified as being iden-

tical to the first order Taylor series expansion embodied in Eq. (2.11). This supports the conclusion stated in section 2.2, according to which a unified approach to structural optimization consists in solving the original problem embodied in Eqs. (2.1-2.3) as a sequence of linearized problems of the form posed in Eqs. (2.15-2.17).

From the foregoing developments, it appears that the virtual load technique can be considered as a particular procedure for computing the constraint graidents. Introducing virtual load cases b_j for each flexibility constraint the corresponding virtual displacement vectors are evaluated by solving the systems of linear equations

$$K\, q_j = b_j \qquad\qquad j = 1,m \tag{3.13}$$

The flexibility gradients are then computed as follows :

$$\frac{\partial u_{j\ell}}{\partial a_i} = - \frac{c_{ij\ell}}{a_i^2} = - \frac{1}{a_i}\, q_j^T\, K_i\, q_\ell \tag{3.14}$$

By the same operaticn the explicit forms of the flexibility constraints defined by Eq. (3.2) are available. Of course the same arguments hold for the stress components expressed in Eq. (3.4) and their gradients are calculated by introducing virtual loading cases t_k (see Eqs. 3.5 and 3.6), yielding

$$\frac{\partial \sigma_{k\ell}}{\partial a_i} = - \frac{d_{ik\ell}}{a_i^2} = - \frac{1}{a_i}\, q_k^T\, K_i\, q_\ell \tag{3.15}$$

This approach to the evaluation of the constraint gradients requires as many additional virtual loading cases as the number of stress and displacement constraints retained in the linearized problem statement, regardless of the number of design variables and of the number of real loading conditions.

3.1.2. The pseudo-loads technique

On the other hand most of the mathematical programming approaches, such as the approximation concepts method developed by SCHMIT [5-8,18], use the pseudo-loads technique of FOX [31] for computing the gradients of the nodal displacements characterizing the finite element model. The gradients of the stress and displacement constraints are then readily evaluated.

Differentiating the equilibrium equations for the ℓ^{th} load case,

$$K\, q_\ell = g_\ell \tag{3.16}$$

leads to

$$K \frac{\partial q_\ell}{\partial a_i} = - \frac{\partial K}{\partial a_i} q_\ell \tag{3.17}$$

In the bar and membrane elements considered in this chapter the stiffness matrix is linear in the design variables so that

$$\frac{\partial K}{\partial a_i} = \frac{K_i}{a_i} \tag{3.18}$$

where K_i is the stiffness matrix of the i^{th} element (see Eq.1.9). Therefore the nodal displacement gradients can be computed by introducing additional loading cases, called pseudo-load vectors:

$$\tilde{g}_{i\ell} = - \frac{1}{a_i} K_i\, q_\ell \tag{3.19}$$

and solving the systems of linear equations

$$K \frac{\partial q_\ell}{\partial a_i} = \tilde{g}_{i\ell} \qquad \begin{array}{l} i = 1,n \\ \ell = i,c \end{array} \tag{3.20}$$

Recalling the definitions (3.1) of the flexibilities and (3.4)

of the stresses, the gradients of the behavior constraints are evaluated from

$$\frac{\partial u_{j\ell}}{\partial a_i} = b_j^T \frac{\partial q_\ell}{\partial a_i} \tag{3.21}$$

and

$$\frac{\partial \sigma_{k\ell}}{\partial a_i} = t_k^T \frac{\partial q_\ell}{\partial a_i} \tag{3.22}$$

The number of pseudo-load vectors is directly related to the number of design variables and to the number of applied loading conditions, and it is independent of the number of behavior constraints.

3.1.3. Selection of a procedure

The decision as to which procedure should be selected to compute the constraint gradients can be based on a comparison of the total number of additional virtual or pseudo-load cases introduced in the structural reanalysis at each given stage :

- if the virtual load technique is used, the number of additional loading cases is equal to the number of stress and displacement constraints retained in the linearized problem statement for the current stage (see Eq. 3.13) ;
- if the pseudo-loads technique is adopted, the number of additional loading cases is equal to the number of independent design variables times the number of applied loading conditions (see Eq. 3.20).

It should be noted that the conventional optimality criteria methods generally take into account a large number of design variables and a small number of critical flexibility constraints, in which case the virtual load technique is obviously

the most efficient scheme for generating their explicit approximate form (and thus for computing their gradients). On the other hand most of the mathematical programming approaches are restricted to a relatively small number of independent design variables, but need evaluation of the gradients for all the stress and displacement constraints, and the pseudo-loads technique is then more advantageous. In fact these two distinct procedures for computing the constraint gradients constitute one of the main reasons why optimality criteria and mathematical programming approaches have been traditionally opposed in the past.

Despite the apparent opposition between the two procedures, they can be analytically related to each other in a very simple way. Substituting Eq. (3.20) into Eq. (3.21) yields the flexibility gradients resulting from the pseudo-load technique :

$$\frac{\partial u_{j\ell}}{\partial a_i} = b_j^T K^{-1} \tilde{g}_{i\ell} \tag{3.23}$$

Using the definition (3.19) of the pseudo-load vectors it follows that

$$\frac{\partial u_{j\ell}}{\partial a_i} = -\frac{1}{a_i} b_j^T K^{-1} K_i q_\ell \tag{3.24}$$

Finally in view of Eq. (3.13), it is apparent that

$$\frac{\partial u_{j\ell}}{\partial a_i} = -\frac{1}{a_i} q_j^T K_i q_\ell \tag{3.25}$$

which is equivalent to the definition (3.14) of the flexibility gradients derived from the virtual load procedure. Note that the same developments can obviously be made for the stresses by replacing (b_j, q_j) with (t_k, q_k).

3.2. Conventional, generalized and hybrid optimality criteria

The generalized optimality criterion (GOC) discussed in section 2.4 as well as the mixed method and the approximation conceptsapproach (section 2.3), use a linearized expression for each behavior constraint. However, in many conventional optimality criteria (COC), such as those reported in Refs. [21,22,24,28,29], only the displacement constraints are replaced with first order explicit approximation while the stress constraints are treated using the classical "Fully Stressed Design" (FSD) concept. In this approach the stress constraints embodied in Eq. (1.4) are transformed into simple side constraints :

$$a_i \geqslant \underline{\tilde{a}}_i \tag{3.26}$$

The minimum values $\underline{\tilde{a}}_i$ are given by the well known stress ratio formula :

$$\underline{\tilde{a}}_i = a_i^o \max_{\ell=1,c} \left| \frac{\sigma^o_{i\ell}}{\bar{\sigma}_i} \right| \tag{3.27}$$

where a_i^o denote the design variables at the current stage and $\sigma^o_{i\ell}$ the corresponding stresses.

As shown in Ref. [31] the FSD procedure can be interpreted as using zero order approximation of the stresses on the scaling line, because it relies on explicit expressions that preserve only the value of the stresses along that line, and not their derivatives. Geometrically, the GOC approximates the stress constraint surfaces by tangent surfaces, while the FSD procedure uses planes normal to the axes of the design space. In both approaches each approximate restraint surface passes by the point of intersection of the scaling line with the corresponding real restraint surface. A graphical interpretation of the differences between conventional (COC) and generalized (GOC) optimality criteria is illustrated in Fig. 3 . The analysis point is denoted a^o,

the solutions of the approximate problems, a^1, and the solution of the real problem, a^*. This one lies at the intersection of the two restraint surfaces, namely, a stress limitation in element 2 and a flexibility constraint. In the GOC each approximate restraint surface is tangent to the real one at its point of intersection with the scaling line. In the COC this feature applies only to the flexibility constraint $u \leq \bar{u}$, while the stress limitation $\sigma_2 \leq \bar{\sigma}_2$ is represented in the design space by a plane $a_2 = \tilde{a}_2$ normal to the a_2 axis.

From a computational point of view, recoursing to the FSD procedure offers two important advantages. First, when the virtual load technique is employed to compute the constraint gradients, the number of addtional loading cases is substantially reduced, because the zero order approximations of the stress constraints de not require any virtual load cases (see Eqs. 3.26, 3.27). Secondly the number of behavior constraints retained in each approximate problem statement (see Eq. 2.19) is also significantly reduced, because all the stress constraints are now replaced with side constraints. This feature is especially beneficial when a dual solution scheme is used to solve the explicit problem, since the dimensionality of the dual problem is equal to the number m of linearized constraints embodied in Eq. (2.19). On the other hand, it is well known that the FSD procedure does not always converge to the true optimum and sometimes leads to instability or divergence of the optimization process.

So we have the choice between the GOC approach, which is rigorous but unefficient from an economical point of view, and the COC approach, which is computationally inexpensive but sometimes unreliable. In order to reap advantage from both zero and first order approximation concepts, a hybrid optimality criterion (HOC) was developed in Refs. [1,31]. In the HOC approach only a small number of stress constraints are treated using the virtual

load procedure and the others are transformed into side constraints using the stress ratio formula. The selection of constraints requiring first order approximation can be made in advance on the basis of the physical judgement of the designer. It can also be performed automatically depending on adequate selection criteria.

A simple selection criterion consists in retaining only the potentially active stress constraints as candidate for first order approximation. Potentially active constraints are defined as those which are close to their limiting values at the current design point :

$$\sigma_{i\ell} \simeq \bar{\sigma}_i \tag{3.28}$$

A second selection criterion states that a stress constraint can be treated by FSD with sufficient accuracy provided

$$\frac{d_{ii\ell}}{a_i} \simeq \sigma_{i\ell} \tag{3.29}$$

that is, if the contribution to the stress $\sigma_{i\ell}$ comes principally from the i^{th} member itself. This condition arises from the fact that the first order approximation (3.5) reduces to the zero order one (3.26) in a statically determinate structure. In case of structural redundancy both approximations are thus close to each other if condition (3.29) is satisfied. Geometrically this occurs when the relevant stress constraint for the i^{th} design variable is represented in the design space by a surface that is roughly parallel to the i^{th} base plane.

Both selection criteria (3.28) and (3.29) must of course be repeated after each structural reanalysis. They must be employed within some adequate tolerances specified by the user. Depending upon the severity of these tolerances, the number of first order approximated stress constraints will be small or large and the HOC approach is accordingly close the COC or to the GOC approaches.

A two-dimensional geometrical interpretation of the COC, GOC and HOC approaches is given in Fig. 4, in the space of the reciprocal design variables. The exact stress constraints $\sigma_1 = \bar{\sigma}_1$ and $\sigma_2 = \bar{\sigma}_2$ are indicated, as well as their linear approximations $\tilde{\sigma}_{1,L} = \bar{\sigma}_1$ and $\tilde{\sigma}_{2,L} = \bar{\sigma}_2$ tangent at T_1 and T_2 and their zero order or FSD approximations $\tilde{\sigma}_{1,F} = \bar{\sigma}_1$ and $\tilde{\sigma}_{2,F} = \bar{\sigma}_2$. From an analysis point x_o, solution of the linearized problem yields the point G, which after restoration on the true constraint surface comes in G' (GOC approach). The exact solution lies in R. The FSD approximation leads to point C and after scaling to point C' (COC approach). Finally employing the first order approximation for σ_1 and the zero order one for σ_2, generates the point H and after scaling the point H' (HOC approach).

3.3. Thermal and body forces

In this section the concepts developed in Ref. [1] are extended to take into account applied loads that depend linearly on the design variables, for example, thermal loads, gravity field loads, etc... The static structural response is then governed by the relations

$$Kq = g = g_c + g_v \tag{3.30}$$

and

$$\sigma = Tq - \sigma_o \tag{3.31}$$

In these expressions it is assumed that the loads g can be split in two contributions, namely, the external loads g_c, which are independent of the design variables (the subscript c stands for "constant") , and the "variable" loads g_v, which are given by :

$$g_v = \sum_{i=1}^{n} a_i \bar{g}_{vi} \tag{3.32}$$

where $\bar{g}_{vi}$ denote constant vectors. The initial stresses σ_o are assumed to be due to thermal effects acting without any other external or internal loads (residual stresses). Note that the index ℓ is omitted in this section, however, the results presented can readily be extended to take into account multiple load conditions.

3.3.1. The scaling concept

The key to being able to construct consistent first and zero order explicit approximations of the behavior constraints again lies in the concept of scaling. If a^1 denotes a point in the design space, the scaling by a factor f yields a new point a^2

$$a_i^2 = f\, a_i^1 \tag{3.33}$$

at which the displacements and stresses become

$$q^2 = \frac{q_c^1}{f} + q_v \tag{3.34}$$

and

$$\sigma^2 = \frac{\sigma_c^1}{f} + \sigma_v - \sigma_o \tag{3.35}$$

where

$$q_c = K^{-1} g_c \qquad \sigma_c = T\, q_c \tag{3.36}$$

$$q_v = K^{-1} g_v \qquad \sigma_v = T\, q_v \tag{3.37}$$

are such that

$$q = q_c + q_v \qquad \sigma = \sigma_c + \sigma_v - \sigma_o \tag{3.38}$$

The foregoing expressions are apparent form the fact that the displacements q_v associated with the variable loads g_v

are not effected by scaling, while the displacements q_c due to the external loads g_c must be divided by the scaling factor (see Ref. [1], section 3.1). The same arguments hold for the stresses ; only the contribution σ_c is modified when scaling is performed.

Considering the flexibility constraints (1.6,3.1) and the stress constraints (1.4,3.4) it can be seen that the scaling factor necessary to bring the design point back on the restraint surfaces are given respectively by :

$$f_{u_j} = \frac{u_{cj}}{\bar{u}_j - u_{vj}} \tag{3.39}$$

and

$$f_{\sigma_k} = \frac{\sigma_{ck}}{\bar{\sigma}_k - \sigma_{vk} + \sigma_{ok}} \tag{3.40}$$

where it is understood that

$$\left.\begin{aligned} u_{cj} &= b_j^T q_c \\ u_{vj} &= b_j^T q_v \end{aligned}\right\} u_j = u_{cj} + u_{vj} = b_j^T q \tag{3.41}$$

$$\left.\begin{aligned} \sigma_{ck} &= t_k^T q_c \\ \sigma_{vk} &= t_k^T q_v \end{aligned}\right\} \sigma_k = \sigma_{ck} + \sigma_{vk} - \sigma_{ok} = t_k^T q - \sigma_{ok} \tag{3.42}$$

3.3.2. <u>First order approximation</u>

In order to construct first order explicit approximation of the behavior constraints, it is necessary to determine analytically their gradients. For that purpose the pseudo-loads technique discussed in section 3.1.2 will be employed. Differen-

tiating the equilibrium equations (3.30) furnishes the first partial derivatives of the nodal displacements as the solution of the linear system :

$$K \frac{\partial q}{\partial a_i} = \frac{\partial g}{\partial a_i} - \frac{\partial K}{\partial a_i} q \tag{3.43}$$

Since the stiffness matrix and the load vector are both linear in the design variables (see Eqs. 1.9 and 3.32), it follows that

$$\frac{\partial K}{\partial a_i} = \frac{K_i}{a_i} \tag{3.44}$$

and

$$\frac{\partial g}{\partial a_i} = \frac{g_{vi}}{a_i} \tag{3.45}$$

so that by solving Eq. (3.43) :

$$\frac{\partial q}{\partial a_i} = -\frac{1}{a_i} K^{-1}(K_i q - g_{vi}) \tag{3.46}$$

Therefore the gradients of the flexibility constraints embodied in Eq. (1.6) are given by

$$\frac{\partial u_j}{\partial a_i} = -\frac{1}{a_i} b_j^T K^{-1}(K_i q - g_{vi}) \tag{3.47}$$

They can be obtained by introducing the pseudo-load vectors

$$\tilde{g}_i = -\frac{1}{a_i} (K_i q - g_{vi}) = -\frac{1}{a_i} (g_i - g_{vi}) \tag{3.48}$$

in the structural analysis phase. Note that the g_i's represent the loads associated with the i^{th} element.

At this point it is important to notice that the virtual load technique can still be employed to evaluate the gradients of the flexibilities. Using again virtual load cases given numerically by the vectors b_j, it appears from Eq. (3.47) that

$$\frac{\partial u_j}{\partial a_i} = - \frac{c_{ij}}{a_i^2} \tag{3.49}$$

with

$$c_{ij} = (q^T K_i q_j - g_{vi}^T q_j) a_i = q_j^T (g_i - g_{vi}) a_i \tag{3.50}$$

where q_j denotes the virtual displacement vector just as in section 3.1.1. It is evident from Eq. (3.49) that the coefficients c_{ij} continue to be equal to the gradients of the flexibilities with respect to the reciprocal design variables. Therefore the first order explicit approximation of the flexibilities can now be constructed by using first order Taylor series expansion :

$$\tilde{u}_j = u_j^o + \sum_{i=1}^{n} \left(\frac{\partial u_j}{\partial x_i}\right)^o (x_i - x_i^o) \tag{3.51}$$

which reduces to

$$\tilde{u}_j = u_j^o - \sum_{i=1}^{n} c_{ij}^o x_i^o + \sum_{i=1}^{n} c_{ij}^o x_i \tag{3.52}$$

In view of Eq. (3.50) it appears that, in a general way :

$$\sum_{i=1}^{n} c_{ij} x_i = q^T K q_j - g_v^T q_j = u_j - g_v^T q_j \tag{3.53}$$

Note that the definition of the flexibility u_j has been used is this equation :

$$u_j = b_j^T q = q_j^T K q = q^T K q_j \tag{3.54}$$

Finally, the first order explicit approximation of a flexibility constraint turns out to be

$$\tilde{u}_j = u_{vj} + \sum_{i=1}^{n} c_{ij} x_i \tag{3.55}$$

where the uppercript ^{o} is now omitted and u_{vj} is defined as the virtual work of the variable loads g_v on the virtual displacements q_j :

$$u_{vj} = g_v^T q_j = b_j^T q_v \tag{3.56}$$

Or course the same developments can be pursued for the stress constraints, yielding the first order explicit approximations

$$\tilde{\sigma}_k = \sigma_{vk} - \sigma_{ok} + \sum_{i=1}^{n} d_{ik} x_i \tag{3.57}$$

with

$$\sigma_{vk} = g_v^T q_k = t_k^T q_v \tag{3.58}$$

and

$$d_{ik} = (q^T K_i q_k - g_{vi}^T q_k)\ a_i = q_k^T (g_i - g_{vi})\ a_i \tag{3.59}$$

where the virtual displacement vector q_k is due to a virtual load case given by t_k.

With regard to the selection of either the pseudo-loads or the virtual load technique for computing the constraint gradients, the arguments provided in section 3.1.3 are still valid. It should be emphasized that again the virtual load procedure helps understanding the nature of the approximation made in linearizing the constraints. Indeed it is apparent from Eqs. (3.55 and 3.57) that the points of intersection of the scaling line with the approximate restraint surfaces $\tilde{u}_j = \bar{u}_j$ and $\tilde{\sigma}_k = \bar{\sigma}_k$ are precisely given by the scaling factors f_{u_j} and f_{σ_k} obtained in Eqs. (3.39 and 3.40) for the real restraint surfaces. This showns that Eqs. (3.55,3.57) continue to represent in the reciprocal design variable space the tangent planes to the real restraint surfaces at their points of intersection with the scaling line. In fact, the quantities c_{ij}, u_{vj}, d_{ik} and $(\sigma_{vk} - \sigma_{ok})$ defining the first order explicit approximations of the flexibility and stress constraints (see Eqs. 3.55 and 3.57), are constant in the case of a statically determinate structure (the explicit problem statement is then

exact). For a statically indeterminate structure, they depend implicitly on the design variables, however, they remain constant along the scaling line.

3.3.3. Zero order approximation

As previously defined, zero order explicit approximation of the stress constraints is obtained by applying the FSD criterion (see section 3.2). It corresponds to replacing the stress constraints with simple side constraints resulting form the stress ratio formula (see Eqs. 3.26, 3.27). In the present situation, where part of the loads depends linearly on the design variables, it is recommended to derive the zero order approximation from the first order one, by assuming that

$$d_{ik} = \sigma_{ci} \, a_i \, \delta_{ik} \tag{3.60}$$

where δ_{ik} denotes the kronecker delta. Eq. (3.57) then reduces to

$$\tilde{\sigma}_i = \sigma^o_{vi} - \sigma_{oi} + \sigma^o_{ci} \frac{a^o_i}{a_i} \tag{3.61}$$

where the quantities with the upperscript ^{o} are frozen to their values at the analysis point. The stress constraints are therefore transformed into the side constraints (3.26) with

$$\tilde{\underline{a}}_i = a^o_i \frac{\sigma^o_{ci}}{\bar{\sigma}_i - \sigma^o_{vi} - \sigma_{oi}} \tag{3.62}$$

With this new definition of the stress ratio formula, the approximate constraint surface $a_i = \tilde{\underline{a}}_i$ is still represented by a plane normal to the i^{th} axis in the design space and passing through the point of intersection of the scaling line with the true constraint surface $\sigma_i = \bar{\sigma}_i$.

It should be clearly recognized that the usual stress ratio formula expressed in Eq. (3.27) for multiple load cases, take the form

$$\tilde{\underline{a}}_i = a_i^o \frac{\sigma_{ci}^o + \sigma_{vi}^o - \sigma_{oi}}{\bar{\sigma}_i} \tag{3.63}$$

This formula is equivalent to Eq. (3.62) only in the case of a statically determinate structure subject to mechanical and thermal loads, because of the following identity :

$$\sigma_{vi} - \sigma_{oi} \equiv 0 \tag{3.64}$$

From a practical point of view, the conventional stress ratio algorithm (3.63) is simpler to employ than the modified one (3.62), which demands separate evaluation of the pure mechanical stresses σ_{ci} and the other contributions $(\sigma_{oi} - \sigma_{vi})$.

4. DYNAMIC CONSTRAINTS ON NATURAL FREQUENCIES

4.1. Problem statement

The problem considered in this chapter consists in minimizing the structural weight while prescribing lower and upper limits on the natural frequencies. In addition side constraints are assigned to the design variables, which are still taken as the transverse sizes of the structural members. Therefore the mathematical programming problem to be solved reads as follows :

$$\text{minimize} \quad W = \sum_{i=1}^{n} \rho_i \, \ell_i \, a_i \tag{4.1}$$

subject to

$$\bar{\omega}_j^2 \geqslant \omega_j^2 \geqslant \underline{\omega}_j^2 \qquad j = 1, m \tag{4.2}$$

$$\bar{a}_i \geqslant a_i \geqslant \underline{a}_i \qquad i = 1, n \tag{4.3}$$

The dynamic constraints expressed in Eq. (4.2) are directly written in terms of the *squares of the frequencies*, because these quantities naturally appear in the eigenproblem characterizing the structural modal analysis :

$$Kq_j - \omega_j^2 \, M \, q_j = 0 \tag{4.4}$$

In this equation, K and M respectively represent the stiffness and mass matrices and (q_j, j = 1, m) are the modal displacements, i.e., the eigenvectors solution of Eq. (4.4), associated with eigenvalues ω_j^2.

Several approaches have been proposed to solve the structural optimization problem embodied in Eqs. (4.1 - 4.3). Analytical methods were employed in some specific, simple situations where the problem can be stated in the form of differential equations. NIORDSON [33] considers transverse vibrations of a beam simply supported at both ends. Using variation calculus with a lagrangian multiplier, he maximizes the fundamental frequency for a given structural weight. Recoursing to a similar analytical method,

TURNER [34] studies axial vibrations of a rod supporting a fixed mass. The problem is formulated as the minimization of the weight for a given frequency. TAYLOR [35] reconsidered the same problem by employing an "energy" approach similar to that of PRAGER [36] for static constraints. He shows that the problem of maximizing the frequency for a given weight yields a solution that also renders minimum the weight of the rod for a given frequency. In a subsequent work, TAYLOR [37] introduced a lower limit on the rod cross-sectional area. PRAGER and TAYLOR [38] give necessary and sufficient optimality conditions for achieving maximum frequency at given weight. They restrict themselves to sandwich structures with continuously varying mass and stiffness properties. Using a similar energy approach SHEU [39] examines one-dimensional structures such as bars and beams, in the numerically interesting case where mass and stiffness are piecewise constant.

The first applications of mathematical programming techniques to problems involving frequency constraints were made by ZARGHAMEE [40], who employed the Rosen gradient projection method for maximizing the fundamental frequency while keeping constant the structural weight (linear constraint). Well suited to complex structural systems analyzed by finite elements, this design procedure needs evaluation of the fundamental frequency gradient. The expressions developed by ZARGHAMEE [40] are exact and explicit (see next sections). The same equations were also derived by FOX and KAPPOOR [32,41] and by ROGERS [42]. They can be traced back to the works of WITTRICK [43] and in some sense, to the work of FRAEIJS de VEUBEKE [44]. RUBIN [45] established a two phase procedure consisting in maximizing the fundamental frequency at constant weight, and then minimizing the structural weight at constant frequency. This second phase is accomplished as long as the fundamental frequency remains close enough to its limiting value. Note that this method cannot easily be extended to the case where multiple frequency constraints are prescribed. In a more general approach, FOX and KAPPOOR [46] make use of the Zoutendijk feasible direction method in order to minimize the structural weight subject to various dynamic constraints. In addition to natural frequency constraints, dynamic response cons-

traints can be treated. ROMSTAD [47] uses the method of approximation programming of GRIFFITH and STEWART [48]. Although restricted to a single constraint in the applications presented, ROMSTAD's formulation can readily be extended to multiple constraints involving several natural frequencies. HAUG, PAN and STREETER [49] propose a multi-purpose method, which is capable of dealing with several types of constraints, including frequency constraints. They employ a steepest descent method in conjunction with a linearization process for the governing equations.

The concept of optimality criterion for dynamic problems appears the first time in the work of YOUNG and CHRISTIANSEN [50]. They design a space truss by assuming that optimality is achieved provided the fully stressed design criterion is satisfied when normalized modal displacements are prescribed. PRAGER and TAYLOR [38] introduced an analytical optimality criterion for continuous sandwich structures. No attempt is made of deriving redesign formulas for discretized systems. VENKAYYA, KHOT and BERKE [51] apply their optimality criteria techniques, initially conceived for static constraints, to the case of frequency constraints. In the examples presented the recursive redesign formulas are restricted to problems involving only one frequency constraint. VENKAYYA and KHOT [52] subsequently included constraints on dynamic stresses and displacements. TAIG and KERR [24], in an innovative approach precursor of dual methods, considered problems with several frequency constraints. Their optimality criteria technique is successfully applied to realistic structural models involving large numbers of elements. KAMAT and SIMITSES [53] studied beam models lying on a continuous elastic foundation and on elastic end supports. They propose an iterative redesign procedure similar to that used in conventional optimality criteria for static constraints. The problem statement consists in maximizing the fundamental frequency for a given weight of the beam. Numerous applications are presented, which differ from each other according to the support conditions and the dead mass distribution.

In the present work the basic concepts established in Ref. [1] and recalled in the previous chapters will be followed, that is, the original problem stated in Eqs. (4.1 - 4.3) will be transformed into

a sequence of explicit problems. The explicit problem statement results from linearizing the frequency constraints with respect to the reciprocal design variables. Solving partially each linearized problem using a primal solution scheme yields the mixed method. Solving exactly each linearized problem using a dual solution scheme leads to the generalized optimality criteria approach.

4.2. Application of the mixed method concept

As in the case of static stress and displacement constraints considered in Ref. [1], we still restrict ourselves to structural models made up of bar and membrane elements. Therefore the stiffness and mass matrices can be written

$$K = \sum_{i=1}^{n} a_i \bar{K}_i \tag{4.5}$$

and

$$M = \sum_{i=1}^{n} a_i \bar{M}_i + M_c \tag{4.6}$$

where $\bar{K}_i$, $\bar{M}_i$ and M_c are independent of the design variables a_i. $\bar{K}_i$ and $\bar{M}_i$ denote respectively the stiffness and mass matrices of the i^{th} element when $a_i = 1$. M_c represents the contribution of the non-structural masses, such as equipments, fuel, etc...

In order to implement the mixed method, consideration must first be given to analytically expressing the gradients of the frequency constraints. Differentiating Eq. (4.4) with respect to the design variables yields

$$\left(\frac{\partial K}{\partial a_i} - \omega_j^2 \frac{\partial M}{\partial a_i} - \frac{\partial \omega_j^2}{\partial a_i} M\right) q_j + (K - \omega_j^2 M) \frac{\partial q_j}{\partial a_i} = 0 \tag{4.7}$$

Premultiplying this equation by q_j^T and taking account of the symmetry of the K and M matrices furnishes :

$$q_j^T M q_j \frac{\partial \omega_j^2}{\partial a_i} = q_j^T \left(\frac{\partial K}{\partial a_i} - \omega_j^2 \frac{\partial M}{\partial a_i}\right) q_j \tag{4.8}$$

Since, in view of Eqs. (4.5) and (4.6),

$$\frac{\partial K}{\partial a_i} = \bar{K}_i \tag{4.9}$$

and

$$\frac{\partial M}{\partial a_i} = \bar{M}_i \quad , \tag{4.10}$$

one finally obtains the gradients under the form :

$$\frac{\partial \omega_j^2}{\partial a_i} = \frac{1}{q_j^T M q_j} \, q_j^T \, (\bar{K}_i - \omega_j^2 \, \bar{M}_i) \, q_j \tag{4.11}$$

In order to facilitate subsequent developments, let us introduce the following notations :

$$k_{ij} = (q_{ij}^T \, K_i \, q_{ij}) a_i \tag{4.12}$$

$$m_{ij} = (q_{ij}^T \, M_i \, q_{ij}) a_i \tag{4.13}$$

$$\mu_j = q_j^T \, M \, q_j \tag{4.14}$$

$$\bar{m}_j = q_j^T \, M_c \, q_j = \mu_j - \sum_{i=1}^{n} \frac{m_{ij}}{a_i} \tag{4.15}$$

where q_{ij} contains the components of the j^{th} eigenmode associated with the i^{th} element ; K_i and M_i are the element matrices, whose dimension equals the number of degrees of freedom for the i^{th} element. The gradients obtained in Eq. (4.11) can then be written in the compact form

$$\frac{\partial \omega_j^2}{\partial a_i} = \frac{k_{ij} - \omega_j^2 \, m_{ij}}{\mu_j \, a_i^2} \tag{4.16}$$

The gradients are numerically available as soon as the modal analysis has been performed, yielding the natural frequencies ω_j and the corresponding eigenmodes q_j.

The gradients of the constraints with respect to the reciprocal design variables

$$x_i = \frac{1}{a_i} \quad , \tag{4.17}$$

which are of primary interest in this study, are expressed as

$$\frac{\partial \omega_j^2}{\partial x_i} = c_{ij} = - \frac{k_{ij} - \omega_j^2 m_{ij}}{\mu_j} \tag{4.18}$$

The explicit forms of the quantities ω_j^2 can now be obtained using Taylor series expansion around the design point x° where the modal analysis is made :

$$\omega_j^2 = \omega_j^{\circ 2} + \sum_{i=1}^{n} \left(\frac{\partial \omega_j^2}{\partial x_i}\right)^{o} (x_i - x_i^{\circ}) + 0 \left[(x_i - x_i^{\circ})^2 \right] \tag{4.19}$$

Restricting the expansion to the first order, the linearized eigenvalues read as follows :

$$\tilde{\omega}_j^2 = \omega_j^{\circ 2} + \sum_{i=1}^{n} c_{ij}^{\circ} (x_i - x_i^{\circ}) \tag{4.20}$$

where, for sake of convenience, the c_{ij}'s denote the gradients of the eigenvalues [see Eq. (4.18)] .

Taking account of the definitions (4.12) and (4.13), it can be observed that, in a general way

$$\sum_{i=1}^{n} k_{ij} x_i = \mu_j \omega_j^2 \tag{4.21}$$

and

$$\sum_{i=1}^{n} m_{ij} x_i = \mu_j - \bar{m}_j \tag{4.22}$$

so that, using Eq. (4.18) :

$$\sum_{i=1}^{n} c_{ij} x_i = - \omega_j^2 + \frac{\omega_j^2}{\mu_j} (\mu_j - \bar{m}_j) = - \omega_j^2 \frac{\bar{m}_j}{\mu_j} \tag{4.23}$$

Finally the Taylor series expansion given in Eq. (4.20) becomes

$$\tilde{\omega}_j^2 = \omega_j^{\circ 2} \left(1 + \frac{\bar{m}_j^{\circ}}{\mu_j^{\circ}}\right) + \sum_{i=1}^{n} c_{ij}^{\circ} x_i \tag{4.24}$$

From the foregoing developments, it appears that the linearized forms of the constraints embodied in Eq. (4.2) are expressed as :

$$\underline{d}_j^\circ \leq \sum_{i=1}^{n} c_{ij}^\circ \, x_i \leq \bar{d}_j^\circ \tag{4.25}$$

with

$$\underline{d}_j^\circ = \underline{\omega}_j^2 - \omega_j^{\circ 2}\left(1 + \frac{\bar{m}_j^\circ}{\mu_j^\circ}\right) \tag{4.26}$$

and

$$\bar{d}_j^\circ = \bar{\omega}_j^2 - \omega_j^{\circ 2}\left(1 + \frac{\bar{m}_j^\circ}{\mu_j^\circ}\right) \tag{4.27}$$

Therefore the linearized problem upon which is based the mixed method can be generated, by replacing the real constraints (4.2) by their linearized forms (4.25) :

$$\text{minimize} \qquad W = \sum_{i=1}^{n} \frac{\ell_i \ell_i}{x_i} \tag{4.28}$$

subject to

$$\underline{d}_j \leq \sum_{i=1}^{n} c_{ij} \, x_i \leq \bar{d}_j \qquad j = 1, m \tag{4.29}$$

$$\underline{x}_i \leq \quad x_i \quad \leq \bar{x}_i \qquad i = 1, n \tag{4.30}$$

Note that the problem is defined in terms of the reciprocal design variables and that the index $^\circ$ is now omitted.

The linearized problem stated in Eqs. (4.28-4.30) exhibits the same characteristics as the problem L constructed in chapter 3 for the case of static constraints on stresses and displacements. Therefore the concept of mixed method can be applied, which consists in partially solving problem (4.28-4.30) using a primal projection method. As explained in section 2.2, the convergence control parameter $\bar{k}$, which is the number of one-dimensional minimizations performed at each stage, must be chosen according to

the degree of nonlinearity that the frequency constraints exhibit in the reciprocal design space. The more non linear the constraints are, the smaller the parameter $\bar{k}$ must be adopted. It should be emphasized, however, that even in the limiting case $\bar{k} = 1$, the mixed method does no longer behave as a pure primal minimization method applied to the real problem. Indeed the restoration phase using scaling of the design variables does not lead necessarily to a feasible boundary point as it is the case when stress and displacement constraints are considered.

The effect of scaling depends now upon mass distribution in the structure. According to the amount of non structural mass, scaling indeed produces different results. The effect of scaling in best investigated by examining the Rayleigh quotient

$$\omega_j^2 = \frac{q_j^T K q_j}{q_j^T M q_j} \tag{4.31}$$

and considering the two following special cases.

(1) There is no fixed masses : the structural mass constitutes then the only contribution to the mass matrix, i.e., $M_c=0$ in Eq. (4.6). Since the mass and stiffness matrices are both linear and homogeneous in the design variables [see Eqs. (4.5) and (4.6)], it is clear that scaling does not modify the eigenvalues nor the associated eigenmodes. Note also that both the numerator and the denominator in the Rayleigh quotient (4.31) are multiplied by the scaling factor, so that each frequency ω_j remains unchanged.

(2) The contribution of the non structural mass is predominating and therefore the structural mass can be neglected, i.e. Eq. (4.6) reduces to $M = M_c$. Looking at the Rayleigh quotient (4.31) it can be seen that the eigenvalues ω_j^2 increase in proportion to the scaling factor. The associated eigenmodes are also modified and must be reevaluated.

In the intermediate case where structural and non structural masses contribute with the same order of magnitude to the mass matrix, the scaling process requires a new complete finite element analysis for determining the modified frequencies and eigenmodes.

In this connection it should be noted that, in view of the numerical results obtained up to now, important non structural masses seem to have a beneficial effect on convergence stability in the overall optimization process. This can be attributed to the fact that in the limiting case (2) where non structural masses are dominant, the terms $\omega_j^2 m_{ij}$ appearing in the constraint gradients (Eq. 4.18) are small when compared to the terms k_{ij} (weak kinetic energy). The structural optimization problem then behaves just as in the static case, where the mixed method concept has proven to be fully valid. In the other cases, recourse to reciprocal variables does no longer reduce necessarily the non-linear character of the constraints, which is the fundamental idea in the mixed method. In such a case it might be useful to adopt more explicit approximations for the constraints, such as the second order Taylor series expansions suggested by MIURA and SCHMIT [54]. Note also that when there is no fixed mass [case (1)], the problem is ill-conditioned, because the structural weight can be rendered arbitrarily small without changing the frequency spectrum.

Finally it is worth mentioning that in the SAMCEF program [15], another restoration procedure is provided to replace that of scaling when necessary. The new restoration phase consists in maximizing the most violated frequency constraint while keeping the structural weight constant. This alternative option is called whenever the frequency constraints reveal violated within some given tolerance after completing the minimization phase. The minimization phase still proceeds by partially solving the linearized problem posed in Eqs. (4.28 - 4.30). This process of alternating weight minimization and frequency maximization is based on the method of RUBIN [45] discussed in section 4.1.

4.3. Generalized optimality criterion

The generalized optimality criterion results from writing the KUHN-TUCKER conditions for the linearized problem embodied in Eqs. (4.28 - 4.30). It is apparent that this explicit optimality criterion takes exactly the same form as in the case of static constraints, that is, it is given by Eqs.(2.21–2.25)

A physical interpretation of the generalized optimality criterion is provided by introducing respectively the elastic energy

$$e_{ij} = \frac{1}{2} q_{ij}^{T} K_i q_{ij} \tag{4.32}$$

and the kinetic energy

$$\mu_{ij} = \frac{1}{2} \omega_j^2 q_{ij}^{T} M_i q_{ij} \tag{4.33}$$

in the i^{th} finite element for the j^{th} vibration mode. In view of the definitions (4.12) and (4.13), these energies are given by :

$$e_{ij} = \frac{1}{2} \frac{k_{ij}}{a_i} \tag{4.34}$$

and

$$\mu_{ij} = \frac{1}{2} \omega_j^2 \frac{m_{ij}}{a_i} \tag{4.35}$$

The difference between these two energies characterizes the optimality of each structural member. Indeed the optimality criterion stated in Eq. (2.21) can also be written

$$\sum_{j=1}^{m} r_j \left(\frac{e_{ij}}{\rho_i \ell_i a_i} - \frac{\mu_{ij}}{\rho_i \ell_i a_i}\right) = c^{st} \tag{4.36}$$

where the r_j's are the lagrangian multipliers associated with the frequency constraints.

The criterion reads thus just as in the static case (see section 3.1.1)provided that the concept of virtual strain energy is replaced with that of difference between elastic energy and kinetic energy. In the special case where only one constraint is assigned - e.g. lower bound on the fundamental natural frequency - the optimality criterion states that in the optimal structure the difference between the elastic energy density and the kinetic energy density is the same in each element. Note that this condition only applies to the so-called active design variables (see Eqs.2.21-2.23). This optimality criterion was first proposed by PRAGER and TAYLOR [38] in the context of continuous models (sandwich structures). It was also found out by several reseachers, using various approaches [34,40,45,51].

With regard to the redesign formulas the general procedure previously established in the static case remains entirely applicable. After each structural analysis phase, yielding the frequencies ω_j and the corresponding eigenmodes q_j, the coefficients c_{ij} are computed from Eq. (4.18). The explicit problem posed by Eqs. (4.28 - 4.30) is then formed and it is solved using either primal or dual algorithms (see sections 2.3, 2.4). It should be clearly recognized that solving the explicit problem (4.27 - 4.29) is equivalent to finding the design variables that satisfy the optimality conditions (2.21-2.25).After this optimization phase is completed, a structural reanalysis is performed with the new design variables and the process is repeated until convergence is achieved. It is important to point out that in the context of an optimality criteria approach, dual algorithms are better than primal algorithms, because most of the time only a small number of frequency constraints must be taken into consideration. On the other hand, if it is found necessary to control the convergence of the overall optimization process, primal algorithms are obviously the most appropriate (mixed method).

5. STABILITY CONSTRAINTS ON LINEAR BUCKLING LOADS

5.1. Problem statement

The structural optimization problem for the case of global stability constraints on the linear buckling load factors exhibits a strong analogy with the problem discussed in chapter 4, regarding frequency constraints. The mathematical programming statement of that problem reads indeed as follows

$$\text{minimize} \qquad W = \sum_{i=1}^{n} \rho_i \ell_i a_i \tag{5.1}$$

subject to

$$\bar{\lambda}_j \geq \lambda_j \geq \underline{\lambda}_j \qquad j = 1,m \tag{5.2}$$

$$\bar{a}_i \geq a_i \geq \underline{a}_i \qquad i = 1,n \tag{5.3}$$

Just as the natural frequencies, the critical load factors λ_j are defined through an eigenproblem :

$$K\, q_j - \lambda_j\, K_G\, q_j = 0 \tag{5.4}$$

where K represents the usual stiffness matrix and K_G, the geometrical stiffness matrix ; (q_j , $j = 1,m$) denote the eigenvectors solution of problem (5.4), associated with eigenvalues λ_j. The physical meaning of the q_j's is that of displacements in the j^{th} buckling mode, for a critical load factor λ_j. In the rest of this chapter λ_j will be simply referred to as "buckling load".

The first attempts of optimizing structures subject to stability constraints were made in the context of column problems, with the aim of maximizing the buckling load for a given structural weight. The pioneering work of CLAUSEN [55] was concerned with the optimal shape of a simply supported column ; more recently KELLER [56] reconsidered the same problem. TADJBAKHSH and KELLER [57] extended the results of KELLER [56]

to the case of columns with other support conditions ; they give a proof of optimality for their solution. In their general structural optimization theory, based on energy considerations, TAYLOR [58], and subsequently PRAGER and TAYLOR [38] formulated optimality conditions for problems involving maximization of the critical buckling load. TAYLOR [58] is solely concerned with columns for which the moment of inertia is proportional to the square of the cross-sectional area , while PRAGER and TAYLOR [38] adopt a linear relation between moment of inertia and cross-sectional area (sandwich structures). In this latter work, as well as in the work of TAYLOR and LIU [59], minimal allowable values are assigned to the design variables. Finally the influence of self-weight on the optimum design was examined by KELLER and NIORDSON [60]. HUANG and SHEU [61] also attacked the problem of self-weight in optimal columns.

All the previously mentioned approaches are based upon consideration of a one-dimensional continuous model and they reduce to finding a single variable function. The discretized models, to which the finite element method is applicable, were studied by ZARGHAMEE [62], in the context of a mathematical programming approach. The aim is to minimize the structural weight while imposing a lower bound on the fundamental buckling load. To this end ZARGHAMEE [62] employs a linearization method related to the cutting plane technique.

The optimality criteria approaches are well suited to problems governed by buckling phenomena, because in addition to the usual side constraints, only one behavior constraint has usually to be taken into account (lowest critical load). In this connection, SIMITSES, KAMAT and SMITH [63] propose recursive redesign formulas based on optimality criteria for the case of elastically supported columns subjected to distributed axial loading. The finite element method is employed for analyzing the discretized model. The redesign relations are similar to those derived by BERKE [28] for a single displacement

constraint. KIUSALAAS [64] considers a more general problem where two buckling modes can be simultaneously critical. He employs conventional redesign relations for multiple constraints but he introduces a relaxation factor in order to allow for controlling the convergence of the optimization process. Results are concerned with columns and portal frames. KHOT, VENKAYYA and BERKE [65], following out the general optimality criteria theory previously proposed in Ref. [51], developed a redesign procedure for minimizing the weight of complex structural systems subjected to a single buckling constraint. The method is successfully applied to portal frames and trusses involving up to 132 finite elements. In this work special techniques for controlling the convergence were investigated, such as over-relaxation and generation of intermediate design points. This study was motivated by instability in convergence encountered in some problems with buckling constraints.

5.2. Application of the mixed method concept

In this section the mixed method statement for problems involving buckling constraints will be based upon linearization of the constraints in the reciprocal design variable space, just as in the case of static and dynamic constraints previously described. This requires computation of the critical load gradients, which in turn assumes to be known the geometrical stiffness matrix dependence on the design variables. For the structural models considered herein, the stiffness matrix K is linear in the design variables, as indicated by Eq. (4.5). This is not the case of the geometrical stiffness matrix K_G, for which the explicit dependence with respect to the design variables disappears. The matrix K_G can still be split into the contributions of the structural members

$$K_G = \sum_{i=1}^{n} K_{G_i} \tag{5.5}$$

where K_{G_i} denotes the geometrical stiffness matrix of the i^{th}

element. The element matrix K_{G_i} is related to the initial stress state in that element and therefore it depends implicitly on all the design variables. It should be emphasized that the geometrical stiffness matrix is independent of the design variables for a statically determinate structure.

Differentiating Eq. (5.4) with respect to the design variables yields :

$$\left(\frac{\partial K}{\partial a_i} - \lambda_j \frac{\partial K_G}{\partial a_i} - \frac{\partial \lambda_j}{\partial a_i} K_G\right) q_j + (K - \lambda_j K_G) \frac{\partial q_j}{\partial a_i} = 0 \qquad (5.6)$$

Premultiplied by q_j^T this relation furnishes, just as in the dynamic case (section 4.1) :

$$\frac{\partial \lambda_j}{\partial a_i} = \frac{1}{q_j^T K_G q_j} q_j^T \left(\bar{K}_i - \lambda_j \frac{\partial K_G}{\partial a_i}\right) q_j \qquad (5.7)$$

From Eq. (5.5) it follows that

$$\frac{\partial K_G}{\partial a_i} = \sum_{i=1}^{n} \frac{\partial K_{G_k}}{\partial a_k} \qquad (5.8)$$

In opposition with the static and dynamic cases previously discussed, the derivatives appearing in Eq. (5.8) are not directly available, because the elements of the geometrical stiffness matrices are linear functions of the stresses acting in the prebuckling state. Therefore the evaluation of the derivatives (5.8) requires that of the quantities

$$\frac{\partial \sigma_k}{\partial a_i} = T_k \frac{\partial q_k}{\partial a_i} \qquad (5.9)$$

where T_k denotes the stress matrix of the k^{th} element and q_k are the generalized displacements of the k^{th} element resulting form the static analysis that gives the prebuckling state. The stress derivatives (5.9) can be computed according to the virtual load or pseudo-loads techniques discussed in section 3.1. Note that these techniques always demand introduction of a certain number of additional fictitious loading cases. Each matrix $\frac{\partial K_{Gk}}{\partial a_i}$ is

then constructed in the same way as the corresponding matrix K_{G_k}, by simply replacing the stresses σ_k by their derivatives (5.9).

This constitutes of course a costly procedure, which fortunetely can be avoided provided the terms $\frac{\partial K_G}{\partial a_i}$ appearing in the gradients expressions (5.7) are negligible. This assumption, which is typical of optimality criteria approaches for static constraints, amounts to not taking into account the effects of structural redundancy. Adopting this assumption it follows that the gradients of the critical loads can be written in the form :

$$\frac{\partial \lambda_j}{\partial a_i} = \frac{1}{\mu_{G_j}} \frac{k_{ij}}{a_i^2} \tag{5.10}$$

where the coefficients k_{ij} are given by

$$k_{ij} = (q_{ij}^T K_i q_{ij}) a_i \tag{5.11}$$

and

$$\mu_{G_j} = q_j^T K_G q_j \tag{5.12}$$

From a practical point of view, the gradients are numerically available as soon as the eigenproblem (5.4) has been solved, yielding the critical loads λ_j and the associated buckling modes q_j. A preliminary static analysis must be accomplished to permit evaluation of the geometrical stiffness matrix K_G.

When the now well known change of variables (4.17) is introduced, the gradients of the critical loads become

$$\frac{\partial \lambda_j}{\partial x_i} = c_{ij} = - \frac{k_{ij}}{\mu_{G_j}} \tag{5.13}$$

where the notation c_{ij} is again employed for sake of conciseness. By pursuing the usual linearization process (see sections 2.1, 3.3 and 4.2), it comes

$$\tilde{\lambda}_j = \lambda_j^o + \sum_{i=1}^{n} c_{ij}^o (x_i - x_i^o) \tag{5.14}$$

where x^o denotes again the reciprocal design point at which structural analysis is performed. Taking into account the definitions (5.11) and (5.13), it can be observed that

$$\sum_{i=1}^{n} c_{ij} \, x_i = - \lambda_j \tag{5.15}$$

As a consequence the linearized forms of the critical loads developed in Eq. (5.14) reduce to

$$\tilde{\lambda}_j = 2 \, \lambda_j^o + \sum_{i=1}^{n} c_{ij}^o \, x_i \tag{5.16}$$

The buckling constraints stated in Eq. (5.2), when linearized at the design point x^o, read then as follows :

$$\underline{\lambda}_j - 2 \, \lambda_j^o \leq \sum_{i=1}^{n} c_{ij}^o \, x_i \leq \bar{\lambda}_j - 2 \, \lambda_j^o \tag{5.17}$$

The notion of mixed method is directly applicable to the problem considered in this section. The scaling of the design variables constitutes indeed an exact restoration procedure. As in the static case (i.e. with static stress and displacement constraints), scaling does not demand any structural reanalysis. When all the direct design variables are multiplied by the same number, the critical buckling loads are also multiplied by this scaling factor. However the scaling operation has now an effect upon the value of the coefficients c_{ij}. In opposition with the static case, where the coefficients c_{ij} remain constant along the scaling line, they are here multiplied by the square of the scaling factor. In other words, if

$$a_i^1 = f \, a_i^o \qquad (\text{i.e. } x_i^1 = \frac{x_i^o}{f}) \tag{5.18}$$

then it follows that

$$\lambda_j^1 = f \, \lambda_j^o \tag{5.19}$$

and

$$c_{ij}^1 = f^2 \, c_{ij}^o \tag{5.20}$$

In these equations the upperscript ^{o} stands for the initial design where structural analysis is made and the upperscript ^{1} refers to the scaled design, f denoting the scaling factor.

As a result, in the space of the reciprocal design variables, the linearized expressions given by Eq. (5.16) furnish the exact values of the critical loads and of their gradients only at the design point x^o where the structural analysis is performed. These linearized forms are no longer correct up to the first order at any point along the scaling line passing through x^o, as it was the case for static stresses and displacements. Geometrically it can be seen that the real restraint surfaces $\lambda_j = \underline{\lambda}_j$ and $\lambda_j = \bar{\lambda}_j$ are represented in the reciprocal design variable space by the planes given by Eq. (5.17) (with equality signs). Such a plane does no longer pass through the point of intersection of the corresponding real restraint surface with the scaling line (see Fig. 1).

However, by making use of the expressions (5.19, 5.20) it is possible to relate the critical loads λ_j and their gradients c_{ij} along the scaling line, without having to recourse to additional structural reanalyses. Therefore tangent planes to the restraint surfaces can still be constructed, by linearizing each buckling constraint at its point of intersection with the scaling line, rather than directly at the design point where the structural analysis is accomplished. From Eq. (5.19) it can be concluded that the scaling factors necessary to bring the design point back to the real restraint surfaces $\lambda_j = \underline{\lambda}_j$ and $\lambda_j = \bar{\lambda}_j$ are respectively

$$f(\underline{\lambda}_j) = \frac{\underline{\lambda}_j}{\lambda_j^o} \tag{5.21}$$

and

$$f(\bar{\lambda}_j) = \frac{\bar{\lambda}_j}{\lambda_j^o} \tag{5.22}$$

Consequently in view of the linearized forms (5.16), the equations of the tangent planes to the surfaces $\lambda_j = \underline{\lambda}_j$ and $\lambda_j = \bar{\lambda}_j$ are respectively given by

$$2\,\underline{\lambda}_j + \left(\frac{\underline{\lambda}_j}{\lambda_j^o}\right)^2 \sum_{i=1}^{n} c_{ij}^o\, x_i = \underline{\lambda}_j \tag{5.23}$$

and

$$2\,\bar{\lambda}_j + \left(\frac{\bar{\lambda}_j}{\lambda_j^o}\right)^2 \sum_{i=1}^{n} c_{ij}^o\, x_i = \bar{\lambda}_j \tag{5.24}$$

From the foregoing developments it is easily verified that by setting

$$c_{ij} = -\,c_{ij}^o \left(\frac{\underline{\lambda}_j}{\lambda_j^o}\right)^2 , \tag{5.25}$$

the linearized problem upon which is based the mixed method concept, reads as follow :

$$\text{minimize} \quad W = \sum_{i=1}^{n} \frac{\rho_i \ell_i}{x_i} \tag{5.26}$$

$$\text{subject to} \quad \frac{\underline{\lambda}_j^2}{\bar{\lambda}_j} \leq \sum_{i=1}^{n} c_{ij}\, x_i \leq \underline{\lambda}_j \tag{5.27}$$

$$\underline{x}_i \leq x_i \leq \bar{x}_i \tag{5.28}$$

Because of the way this explicit problem has been generated, the mixed method can be employed exactly like in the static case (section 2.3). However it should be clearly recognized that the explicit expressions embodied in Eq. (5.14) are the true linearized forms of the critical loads only if the structure is statically determinate. Otherwise they contain an error that is related to the degree of structural redundancy. When this error is negligible, or when the coefficients c_{ij} are computed using the exact gradients given in Eq. (5.7), then the linearized problem stated in Eqs. (5.26-5.28) can be geometrically interpreted just as the problem L developed in section 2.1 for the case of stress and displacement constraints. Namely, each real restraint surface

($\lambda_j = \underline{\lambda}_j$ or $\lambda_j = \bar{\lambda}_j$) is replaced in the reciprocal design variable space with its tangent plane at its point of intersection with the scaling line (see Fig. 1).

In these circumstances, the mixed method described in section 2.3 can be effectively used to control the convergence of the optimization process. In the special limiting case where only one minimization step is completed at each stage ($\bar{k} = 1$), the mixed method still reduces to a pure projection method of non linear programming. Recourse to the control parameter $\bar{k}$ should in fact reveal especially useful for optimization problems involving global buckling constraints. Indeed KHOT, VENKAYYA and BERKE [65] report some difficulties in achieving convergence when applying their optimality criteria techniques to such problems. The mixed method, which lies between optimality criteria and mathematical programming approaches, seems capable of overcoming these difficulties. Unfortunetely, in the current version of the SAMCEF program [15], in which the mixed method is available, the convergence control capability has not been tested extensively enough for buckling constraints so as to provide a definitive answer with regard to its efficiency.

5.3 Generalized optimality criterion

As explained in section 2.3, the generalized optimality criteria approach consists in solving exactly at each stage the linearized problem stated in Eqs. (5.26-5.28). Alternatively the optimality criterion can be written explicitly by using the KUHN-TUCKER conditions. Just as in the static and dynamic cases, these conditions take the form expressed in Eqs.(2.21-2.25) with the proper definition of the coefficients c_{ij}. Another form of the optimality criterion, which is more suitable to physical interpretation, reads as follows :

$$\sum_{j=1}^{m} r_j \frac{e_{ij}}{\rho_i \ell_i a_i} = c^{st} \qquad (5.29)$$

where the quantity

$$e_{ij} = \frac{1}{2} \frac{c_{ij}}{a_i} = \frac{1}{2} q_{ij}^T K_i q_{ij} \qquad (5.30)$$

represents the strain energy in the i^{th} element for the j^{th} buckling mode. Note that each lagrangian multiplier r_j must be positive or zero according to whether the associated buckling constraint is active or not.

The criterion possesses an interesting physical interpretation in term of strain energy densities, in a similar way as for the static case (section 3.1.1). In the special but frequent situation where only one buckling constraint is assigned, the optimality criterion states that <u>in the optimal structure, the strain energy density associated with the critical buckling mode is the same in each element</u>. Several researchers[38, 63, 64, 65] have proposed the optimality criterion posed by Eq. (5.29), but without deriving efficient redesign relations.

As previously reported in this work on several occasions, the only rigorous way of finding the design variable values that satisfy the optimality criteria equations is to solve exactly the linearized problem using efficient, special purpose mathematical programming algorithms. So the redesign procedure is as follows : after each structural reanalysis, yielding the coefficients c_{ij} via Eqs. (5.11, 5.13 and 5.25), the new estimates of the optimal design variables are generated by applying primal or dual algorithms to the explicit problem embodied in Eqs. (5.26-5.28). As in the dynamic case considered in chapter 4, dual algorithms are most often recommended, because the number of critical buckling constraints is usually small.

6. EXTENSION TO FLEXURAL MODELS

The structural optimization methods reported in Ref. [1] and in the previous chapters of this work were essentially developed in the context of thin-walled structures for which finite element idealization by means of bars and membranes leads to adequate estimation of the quantities describing the structural response (e.g., stresses, displacements, natural frequencies, critical buckling loads). The stiffness and mass matrices of these structural models are linear in the design variables, which so far were the transverse sizes of the elements. This situation of course facilitates derivation of the mixed method and generalized optimality criterion.

Such a finite element representation is valid only when each structural member can be assumed to be in a plane state of stress. So at the element level the applied forces must act mainly in extension. If the flexural loading cannot be neglected, then it is necessary to introduce other types of finite elements than the bars and membranes considered up to now. The stiffness matrix of a bending element such as a beam or a plate is no longer proportional to its transverse size and therefore the optimization methods proposed in the previous sections must be modified.

6.1. Choice of the design variables

The optimum design of structures for which the stiffness properties do not depend linearly on the design variables was mainly investigated in the case of continuous models. Most often the optimization problem is restricted to constraints on the natural frequencies and on the buckling loads [66,67,68]. In this chapter attention is first focused on discretized models made up of pure beam and plate elements. Next consideration will be given to the more general case where flexion and extension forces are simultaneously applied at the element level.

6.1.1. Beam elements

The way to deal with structural optimization problems for a beam subject to pure bending depends upon the relationship between the principal moment of inertia I and the cross-sectional area a. A wide variety of situations is taken into account by means of the following relation :

$$I = c\ a^p \tag{6.1}$$

where c is a constant that depends only on the shape of the beam cross-section and p is a positive number [53, 64]. Most of the time p is taken as an integer number, equal to 1, 2 or 3.

The case $p = 1$ corresponds to thin-walled beams, for example, sandwich beams, pipes with fixed diameter but variable thickness, etc... The methods described in sections 3, 4 and 5 remain then fully applicable, since the stiffness matrix of each structural member continues to be proportional to the design variable, which can be either the cross-sectional area or the moment of inertia. In the context of discretized models, VENKAYYA [23] was the first to study this kind of problem. He makes the observation that when the ratio of the moment of inertia to the cross-sectional area is kept constant, the scaling concept remains entirely valid. This is an important property from the point of view of optimality criteria approaches

The case $p = 2$ is that of beams with uniformly varying cross-section; the shape of the cross-section is kept constant while its area varies along the beam axis (dilatation or contraction). This type of beam problem was analytically investigated by KELLER [56] and TAYLOR [58], for a single constraint on the critical buckling load (Euler beam).

Finally the case $p = 3$ is concerned with beams with full cross-section whose height varies while other sizes are fixed (case similar to that of plate elements) [53].

In order to illustrate the versatility of relation (6.1), let us consider a beam with rectangular cross-section. It follows that

$$I = \frac{bh^3}{12} \tag{6.2}$$

and

$$a = b\ h \tag{6.3}$$

where b is the width of the cross-section and h, its height. Varying the width b and keeping constant the height h leads to

$$I = \frac{h^2}{12}\ a \tag{6.4}$$

which corresponds to $c = \frac{h^2}{12}$ and $p = 1$ in Eq. (6.1) (linear relation). If now the height h is the variable , it follows that

$$I = \frac{1}{12b^2}\ a^3 \tag{6.5}$$

namely, $c = \frac{1}{12b^2}$ and $p = 3$ in Eq. (6.1) (cubic relation). Finally keeping the ratio h/b constant (constant shape cross-section) yields

$$I = \frac{h}{12b}\ a^2 \tag{6.6}$$

that is, a quadratic relation.

On the other hand, in the case of complex cross-sectional shapes, the number p appearing in Eq. (6.1) can be taken non integer, by recoursing to empirical formulas relating the moment of inertia to the cross-sectional area for standard gage sizes. MOSES and ONADA [69], as well as KAVLIE and MOE [70], employ such formulas for optimizing elastic grids. Note for example the following relation, which is deduced from English

norms for beam construction :

$$I = 1.007 \left(\frac{a}{1.480}\right)^{2.65} \tag{6.7}$$

This expression leads to the values $c = 0.356$ and $p = 2.65$ in the general relation (6.1).

Once the structural model is characterized via an explicit relation of the form given in Eq. (6.1), the problem of choosing appropriate design variables disappears. The natural choice for expressing the weight of a beam is that the main design variable must be the cross-sectional area. However the stiffness properties are primarily dependent on the moment of inertia and therefore it can be expected that the explicit approximations of the behavior constraints will be the most accurate when written in terms of the reciprocal moments of inertia. Since both quantities - cross-sectional area and moment of inertia - are connected to each other through Eq. (6.1), either one can be taken as independent design variable. For a beam subjected to pure bending, the flexural rigidity is proportional to the moment of inertia and therefore in a finite element context the structural stiffness matrix exhibits the following explicit form in terms of the cross-sectional areas :

$$K = \sum_{i=1}^{n} a_i^p \bar{K}_i \qquad p > 0 \tag{6.8}$$

where each matrix $\bar{K}_i$ is independent of the design variables a_i.

6.1.2. Plate elements

With regard to plate elements subjected to pure bending, two cases must be distinguished. The first case is that of sandwich plates with constant core thickness. The sheet thicknesses constitute then the design variables.Therefore the stiffness matrix continues to depend linearly upon the design variables and the methods previously developed in this work remain fully applicable.

The second case is concerned with conventional plates with variable thickness. For this type of structural model the stiffness is proportional to the cube of the thickness and hence the relation (6.8) must be chosen with $p = 3$. In the context of continuous models, plate optimum design was investigated by OLHOFF [71] and ARMAND [72], for the case of a single constraint on the fundamental natural frequency, and by FRAUENTHAL [73], for the case of specified critical buckling load.

6.1.3. Flexion - extension elements

When flexion and extension loadings act simultaneously with comparable intensity at the element level, the definition (6.8) of the stiffness matrix can no longer characterize the structural model with sufficient accuracy. In order to illustrate this fact, let us consider again the beam with rectangular cross-section previously discussed and assume that only the cross-section width can be varied. If the beam is subjected to combined flexion, some terms of the stiffness matrix require $p = 1$ in Eq. (6.1) (principal moment of inertia proportional to the design variable), while others demand $p = 3$ (secondary moment of inertia proportional to the cube of the design variable).

In these circumstances, it is necessary to recourse to a more complete definition of the element stiffness matrices in terms of the design variables. As suggested by KIUSALAAS [64,74], the following expression can be chosen :

$$K_i = \sum_{p=o}^{3} K_i^{(p)} a_i^p \qquad (6.9)$$

where the matrices $K_i^{(p)}$ are independent of the design variables a_i. Each matrice $K_i^{(p)}$ must be memorized independently for each finite element, in such a way that the contributions of the various terms can be subsequently separated.

The expression (6.9) is rather general and it permits great variety in the choice of the element sizes that can be varied. As an example let us consider a beam element with hollow circular cross-section (thick-walled pipe). The extensional stiffness is included in Eq. (6.9) through a term proportional to the design variable, which is taken as the cross-sectional area, while the dependence of the flexural stiffness upon the design variable is related to the way the beam cross-sectional sizes can be varied. So if the cross-section is kept similar to itself all along the length of the tube, then it follows that

$$K_i = K_i^{(1)} a_i + K_i^{(2)} a_i^2 \tag{6.10}$$

If now the external diameter varies, while the internal one is fixed, then another relation must be adopted :

$$K_i = K_i^{(1)} a_i + K_i^{(3)} a_i^3 \tag{6.11}$$

Based on similar arguments, it can be seen that a flat shell element, made up of the assembling of a membrane element (extension) and a plate element (flexion), leads to two distinct contributions to the stiffness in Eq. (6.9) : one is linear in the thickness and the other one is cubic, just as in Eq. (6.11).

On the other hand the mass matrix of each finite element continues to exhibit a linear form in terms of the design variables :

$$M_i = \sum_{p=o}^{1} M_i^{(p)} a_i^p \tag{6.12}$$

By following basically the same procedures as for the case of thin-walled structures (see chapters 3, 4 and 5), it is possible to construct explicit optimality criteria, provided the contributions of the various element matrices $K_i^{(p)}$ (and $M_i^{(p)}$ for frequency constraints) are decomposed in the constraint explicit approximations.

6.2. Application of the mixed method concept

From the foregoing developments it follows that a general automatic redesign procedure can be generated, by following out the now customary scheme for deriving the mixed method and the generalized optimality criterion (see section 3.3, 4.2 and 5.2). In order to outline the procedure followed, attention is restricted to the case of constraints on the static structural response, i.e., upper limits on stresses and displacements. It is assumed that the structural discretization is made up entirely of elements of the same type. Therefore the stiffness matrix exhibits the form given in Eq. (6.8), where p takes on the same value for each member. To fix ideas the method development is limited to displacement constraints, keeping in mind that stress constraints can always be converted into displacement constraints (first order approximation) or replaced with minimum gage constraints (zero order approximation).

Just as in the case of thin-walled structures, the mixed method concept relies on a change of variables tending to reduce the nonlinear character of the constraints :

$$x_i = \frac{1}{a_i^p} \tag{6.13}$$

The next step is to linearize the constraints in terms of the new variables x_i, which requires gradient evaluation (see for example Eq. 2.11). Differentiating the equilibrium equation $K\ q = g$ and using the definition of a flexibility $u_j = b_j^T\ q$ leads to the constraint gradients via the pseudo-loads technique (see section 3.1.2) :

$$\frac{\partial u_j}{\partial x_i} = -\ b_j^T\ K^{-1}\ \frac{\partial K}{\partial x_i}\ q \tag{6.14}$$

Alternatively, introducing the virtual load vectors $q_j = K^{-1}\ b_j$ associated with the flexibilities, the gradients (6.14) can also be written as :

$$\frac{\partial u_j}{\partial x_i} = - q_j^T \frac{\partial K}{\partial x_i} q \tag{6.15}$$

This latter expression corresponds to the definition of the gradients using the virtual load technique (see section 3.1.1).

In terms of the new variables x_i, the global stiffness matrix given in Eq. (6.8) becomes

$$K = \sum_{i=1}^{n} \frac{\bar{K}_i}{x_i} \tag{6.16}$$

Its first partial derivatives are therefore :

$$\frac{\partial K}{\partial x_i} = - \frac{\bar{K}_i}{x_i^2} = - \frac{K_i}{x_i} \tag{6.17}$$

when K_i is the stiffness matrix of the i^{th} element. Finally, by substituting Eq. (6.17) into Eq. (6.15), the flexibility gradients take the form :

$$\frac{\partial u_j}{\partial x_i} = \frac{1}{x_i} q_j^T K_i q \tag{6.18}$$

so that the following identity holds

$$u_j \equiv \sum_{i=1}^{n} \left(\frac{\partial u_j}{\partial x_i}\right) x_i \tag{6.19}$$

By employing the general linearization process outlined in sections 3.3, 4.2 and 5.2, we can express the explicit forms of the flexibilities as :

$$\tilde{u}_j = u_j^o - \sum_{i=1}^{n} c_{ij}^o x_i^o + \sum_{i=1}^{n} c_{ij}^o x_i \tag{6.20}$$

where the coefficients c_{ij}^o denote the flexibility gradients evaluated at the design point x^o :

$$c_{ij}^o = \left(\frac{\partial u_j}{\partial x_i}\right)^o \tag{6.21}$$

In view of Eq. (6.19), it is apparent that the first and second terms in Eq. (6.20) cancel each other, so that the linearized constraints exhibit the same form as in the case of bar and membrane elements previously considered (omitting now the superscript ^{o}) :

$$\tilde{u}_j = \sum_{i=1}^{n} c_{ij}\, x_i \leq \bar{u}_j \qquad j = 1,m \tag{6.22}$$

Therefore the linearized problem statement is similar to that stated in Eqs. (2.15-2.17), except that the objective function must be replaced by :

$$W = \sum_{i=1}^{n} \frac{\rho_i \ell_i}{x_i^{1/p}} \tag{6.23}$$

The linearized forms of the constraints embodied in Eq. (6.22) are still exact up to the first order at any point along the scaling line in the space of the x_i's (generalized reciprocal space), or in the space of the a_i^p. So the scaling process keeps all its properties (see sections 3.1 and 3.2), provided scaling is performed on the quantities a_i^p, rather than directly on the design variables a_i. As a result scaling can still be employed to bring the design point back on the boundary of the feasible region (restoration phase). The mixed method thus behaves exactly as in the case of structural models made up of bar and membrane elements (see section 2.3). The parameter $\bar{k}$ continues to permit control over convergence in the optimization process. In the limiting case $\bar{k} = 1$, the mixed method identifies itself to a primal mathematical programming algorithm.

6.3. The generalized optimality criterion

As explained in section 2.4.2 the explicit form of the generalized optimality criterion is produced by expressing the KUHN-TUCKER conditions for the linearized problem. When recast in terms of the direct design variables a_i, the linearized problem takes the form

minimize $\quad W = \sum_{i=1}^{n} \rho_i \, \ell_i \, a_i$ (6.24)

subject to

$$\sum_{i=1}^{n} \frac{c_{ij}}{a_i^p} \leq \bar{u}_j \qquad j = 1,m \tag{6.25}$$

$$\underline{a}_i \leq a_i \leq \bar{a}_i \qquad i = 1,n \tag{6.26}$$

The optimality criteria equations are very similar to that embodied in Eqs. (2.21-2.25), provided that care is taken of the exponent p appearing in Eq. (6.25). For example the redesign relations for the active design variables must read as follows :

$$a_i = \left(\frac{p}{\rho_i \ell_i} \sum_j c_{ij} \, r_j\right)^{\frac{1}{p+1}} \tag{6.27}$$

where it is understood that the dual variables r_j (i.e. the lagrangian multipliers) must satisfy the complementarity conditions (2.24, 2.25).

To obtain a physical interpretation of the optimality criterion, we introduce the virtual strain energies

$$e_{ij} = q_j^T \, K_i \, q \tag{6.28}$$

which are related to the coefficients c_{ij} via Eqs. (6.21), (6.18) and (6.13) :

$$c_{ij} = (q_j^T \, K_i \, q) \, a_i^p = e_{ij} \, a_i^p \tag{6.29}$$

Therefore the optimality criterion stated in Eq. (6.27) takes again the customary "energetic" form :

$$\sum_j r_j \, \varepsilon_{ij} = c^{st} \tag{6.30}$$

where the ε_{ij}'s are virtual energy densities per unit weight :

$$\varepsilon_{ij} = \frac{e_{ij}}{\rho_i \ell_i a_i} \tag{6.31}$$

Note that the design variables a_i are still defined as the transverse sizes of the structural members, so that $\rho_i \ell_i a_i$ represents the weight of the i^{th} element.

In the special case where only one flexibility constraint is specified, the optimality criterion states that the virtual strain energy density must be the same in each element. The optimality criterion can then be employed to generate analytically the solution to the explicit problem (6.24-6.26). By following out the same calculations as in section 2.2.2 of Ref. [1], the optimum design variables can be expressed explicitly in terms of known quantities. Taking Eq. (6.27) with only one lagrangian multipliers (m=1) yields

$$a_i = \left(r\, p \frac{c_i}{\rho_i \ell_i}\right)^{\frac{1}{p+1}} \qquad i = 1,\tilde{n} \tag{6.32}$$

where $\tilde{n}$ is the number of active design variables. Substituting then this latter equation into the constraint equation

$$\sum_{i=1}^{\tilde{n}} \frac{c_i}{a_i^p} = \bar{u} - u_o \tag{6.33}$$

(where u_o is the contribution of the passive design variables) leads to an explicit expression for the lagrangian multiplier r :

$$(r\, p)^{\frac{p}{p+1}} = \frac{1}{\bar{u} - u_o} \sum_{k=1}^{\tilde{n}} \left[\rho_k\, \ell_k\, c_k^{\frac{1}{p}}\right]^{\frac{p}{p+1}} \tag{6.34}$$

Finally, by reintroducing Eq. (6.34) into Eq. (6.32), it follows that

$$a_i = \left\{\frac{1}{\bar{u} - u_o} \sum_{k=1}^{\tilde{n}} \left[(\rho_k\, \ell_k)^{\frac{p}{p+1}}\, c_k^{\frac{1}{p+1}}\right]\right\}^{\frac{1}{p}} \left(\frac{c_i}{\rho_i \ell_i}\right)^{\frac{1}{p+1}} \qquad i=1,\tilde{n} \tag{6.35}$$

This latter equation gives the solution of the explicit problem provided the subdivision of the design variables into active and

passive groups is correctly defined. It is worthwhile mentioning that Eq. (6.35) is well suited for the design of plates in bending. Since then $p = 3$, the redesign relation (6.35) involves the fourth root of the flexibility coefficients c_i, rather than the third root as employed by ARMAND and LODIER [75] on an intuitive basis.

7. COMPUTER PROGRAM IMPLEMENTATION

This chapter gives some indications about the computer implementation of the concepts developed in the present work. The resulting optimization capabilities are fully integrated in the general purpose finite element program SAMCEF [15], which is applicable to large structural systems (several thousands of degrees of freedom and finite elements). The optimization module of SAMCEF is built to loop on the general static, dynamic and stability analysis modules. This implies that all the possibilites offered by these modules are still available, as well as those of the auxiliary modules like mesh generators, plotting modules for input and output, etc... Given a finite element model, the user may ask for one or more optimization steps, without any thing else to do than to define the constraints.

At present time the elements whose dimensions are taken into account in the operational version of the program are limited to the bars and plates elements in extension. The next version will include bending elements (beams and flat shells) according to the indications given in chapter 6. However the structure may be idealized using any other type of element but their dimensions remain unchanged by the optimization process. In particular the possibility of using super-elements exists and is very useful for representing pre-optimized or fixed parts in the structure. In absence of specifications the thickness or cross-section of each finite element is taken as a design variable and is allowed to be independently resized. However the finite elements can be grouped in such a way that one design variable is assigned to each group. The possibility exists in membrane elements to represent composite materials like reinforced resins as the superposition of a number of layers with independent orthotropic properties. The thickness of each layer is then a separate design variable so that the superposition of results allows for the definition of the composite.

In the current version of the SAMCEF program, three distinct optimization modules are provided for treating separately static constraints on stresses and displacements (see chapter 3), dynamic constraints on natural frequencies (see chapter 4) and stability constraints on linear buckling loads (see chapter 5). It is envisioned that the next version will include the possibility of taking simultaneously into account any combinations of these three types of behavior constraints by using a data base system. The program can evidently make use of the restart capabilities of the analysis modules but it is also designed for an interactive use when the designer examines the solution after each optimization stage. After each structural analysis the program can be stopped, and then automatically restarted without repeating the analysis.

The selection of the optimization algorithm is left to the user but recommendations based on examples are provided to help the choice which will however probably remain problem dependent.

7.1. Fabrication requirements

In practical structural design problems it is necessary to take into account various fabricational limitations. In addition to the usual side constraints, which allow the designer to prescribe realistic minimum or maximum gauge sizes, design variable linking is often necessary to permit generation of a practically meaningful design. Also the computer program must include the possibility of treating fixed members.

In order to insure sufficient accuracy in the structural analysis, finite element models usually involve large numbers of elements but a relatively small number of independent design variables, because of the subdivision of the various members into some preselected groups (see section 1.4). As expressed in Eq. (1.3) design variable linking consists of equality constraints on the member sizes and it can therefore be easily handled in the problem formulation.

Using the convenient notation [74]

$$a_k = a_i \qquad \begin{cases} k \in i \\ i = 1, n' \end{cases} \tag{7.1}$$

where $k \in i$ implies "all elements that are linked to the i^{th} independent design variable", the explicit problem statement takes the form (see Eqs. 2.18-2.20) :

$$\text{minimize} \qquad W = \sum_{i=1}^{n'} \ell'_i \, a_i \tag{7.2}$$

$$\text{subject to} \qquad \sum_{i=1}^{n'} \frac{c'_{ij}}{a_i} \leqslant \bar{u}_j \qquad j = 1, m \tag{7.3}$$

$$\underline{a}_i \leqslant a_i \leqslant \bar{a}_i \qquad i = 1, n' \tag{7.4}$$

with

$$\ell'_i = \sum_{k \in i} \rho_k \ell_k \tag{7.5}$$

and

$$c'_{ij} = \sum_{k \in i} c_{kj} \tag{7.6}$$

Note that ℓ' represents the cumulated weight of all elements in the i^{th} group when $a_i = 1$. When some or all stress constraints are treated using the FSD procedure (zero order approximation, see section 3.2), they are included in the minimum size limits using the modified stress ratio formula

$$\underline{\tilde{a}}_i = a_i^o \max_{\substack{k \in i \\ \ell = 1, c}} \left\{ \frac{\sigma^o_{k\ell}}{\bar{\sigma}_k} \right\} \tag{7.7}$$

From the point of view of the minimization algorithms, no change has to be introduced in the primal and dual solution schemes discussed in chapter 2, because the explicit problem after linking exhibits exactly the same form as the explicit problem

before linking. The only effect of linking is to reduce the number of independent design variables from n to n'.

For various reasons it is sometimes interesting to incorporate in the structural model, finite elements that are not associated with any design variable (superelements, fixed members, etc...). The stiffness and mass properties of such elements are not affected by redesign, however their contributions to the approximate constraints statement must be taken into account in the form of fixed terms to be substracted from the upper bounds $\bar{u}_j$. For example, when the virtual load technique is employed to generate explicit approximations of flexibility constraints, Eqs. (3.7, 3.8) should be rewritten

$$\sum_{i=1}^{n'} \frac{c_{ij}}{a_i} \leq \bar{u}_j - u_{jo} \qquad (7.8)$$

with

$$u_{jo} = \sum_{i>n'} q_j^T K_{io} q \qquad (7.9)$$

where n' is the number of variables and K_{io} are the stiffness matrices of the fixed members (i.e. for $i > n'$).

7.2 Optimization algorithms

Four optimization algorithms are available in the SAMCEF program. The user can select any one of them depending upon the characteristics of each specific problem : the number of independent design variables, the number of behavior constraints and the expected degree of non linearity of the constraints.

7.2.1 PRIMAL 1 optimizer

PRIMAL 1 is a first order projection algorithm based on the well known gradient projection method for linear constraints.

It uses an orthogonal projection operator to generate a sequence of search directions that are constrained to reside in the sub-space defined by the set of active constraint hyperplanes. The successive search directions are conjugated to each other as long as there is no change in the set of active constraint.

The PRIMAL 1 optimizer operates in the space of the reciprocal design variables and it produces a sequence of steadily improved feasible designs with respect to the linearized problem. Hence PRIMAL 1 can be adequately used for seeking a partial solution to each linearized problem, in such a way that the constraints of the primary problem remain almost satisfied. This is achieved by prescribing an upper limit on the number of one-dimensional minimizations performed before updating the approximate problem statement (parameter $\bar{k}$; see section 2.3). PRIMAL 1 is thus the recommended option when the constraints of the primary problem are highly nonlinear in the reciprocal variables (strong structural redundancy). The algorithm is described in detail in section 6.1 of Ref. [1].

7.2.2 PRIMAL 2 optimizer

PRIMAL 2 is a second order projection algorithm especially well suited to the solution of problems with separable objective function and linear constraints. It uses a weighed projection operator to generate a sequence of Newton's search directions in the subspace formed by the intersections of the active constraint hyperplanes. PRIMAL 2 exhibits the same features as PRIMAL 1, but it is far more efficient. Therefore the PRIMAL 2 optimizer is well suited to solve exactly each linearized problem, in which case it produces the same iteration history as the dual methods.

PRIMAL 2 is thus a recommended option when the behavior constraints are very shallow in the space of the reciprocal variables (weak structural redundancy). Note however that the DUAL 2 option is usually more efficient. The PRIMAL 2 algorithm is described in section 6.2 of Ref. [1].

7.2.3 DUAL 2 optimizer

The dual method formulation, which exploits the separable form of the approximate problem, consists in maximizing the explicit dual function subject to nonnegativity constraints on the dual variables. As explained in section 2.4.1, this approach is very efficient, because the dimensionality of the dual space is primarily dependent on the number of critical behavior constraints, which is relatively low for many structural optimization problems of practical interest.

DUAL 2 is a dual method which employs a second order Newton type of algorithm to find the maximum of the dual function when all the design variables are continuous. It operates in a sequence of dual subspaces with gradually increasing dimensions, so that the effective dimensionality of the dual problem does not exceed the number of active behavior constraints by more than one. Since the DUAL 2 optimizer has been found to be highly efficient in practice, it is the recommended option for pure continuous variable problems, unless the number of active behavior constraints is expected to be high. The algorithm is described in section 4.3 of Ref. [1], as well as in section 3 of Ref. [13].

7.2.4 DUAL 1 optimizer

DUAL 1 is a dual method which employs a specially devised first order gradient projection type of algorithm to find the maximum of the dual function when all or some of the design variables are discrete (see Ref. [19]). The DUAL 1 algorithm incorporates special features for handling the dual function gradient discontinuities that arise from the primal discrete variables. These discontinuities occur on specific hyperplanes in the dual space. The DUAL 1 algorithm determines usable search directions by projecting the dual function gradient on the intersection of the successively encountered first order discontinuity planes. It should be noted that the DUAL 1 optimizer remains applicable to

pure continuous variables problems, in which case it reduces to a special form of the conjugate gradient method. However it is generally less efficient than the DUAL 2 optimizer. DUAL 1 was initially conceived for the ACCESS 3 program [14] and it is described in detail in chapter 4 of Ref. [13].

7.2.5 Core requirements

Because of their special implementation, which takes advantage from the simple form of the side constraints, the projection algorithms PRIMAL 1 and PRIMAL 2 require a modest core size (see Ref. [1]). The number of words necessary to solve a given problem, with n independent design variables and n linearized behavior constraints, is given by

$$\text{nbr of words} = n \times (m + 10) + \frac{m \times (m+1)}{2} + 3 \times m \qquad (7.10)$$

This formula also applies to the DUAL 2 optimizer, however it should be modified for the DUAL 1 optimizer when discrete variables are involved. Fig. 5 represents graphically the core requirement given by Eq. (7.10). It should be clearly recognized that, because design variable linking is most often employed in practical applications, the SAMCEF program is capable of dealing with structural optimization problems involving thousands of finite elements. For example a problem with 500 design variables - which might well correspond to 5000 elements - and 20 linearized behavior constraints can fit in a core of less than 16000 words (on IBM 370-158 the computer program requires then 256 K). Note that no core limitation is associated with the analysis modules of SAMCEF, because they are organized in such a way that they can solve very large problems using a modest size central core. They employ a frontal equation solver with substructing and extensive peripheric storage.

It can be seen from examining Fig. 5 that the main limitation arises from the number m of behavior constraints retained in the

linearized problem statement. If the number of constraints is raised up to 100 in the previous example (with 500 design variables), then the central core requirement increases to 60000 words. In addition it is evident that linearizing the behavior constraints demands a large computational effort, because this implies treating additional loading cases in the structural reanalyses (see section 3.1). Finally the computer time expanded in the optimizer itself can become prohibitive when the number of linearized constraints is large. This is apparent in the case of dual algorithms, since the dimensionality of the dual problem is precisely equal to the number of linearized constraints. It is also true in the case of primal algorithms, because most of the computational effort quickly increaseswith the number of linear constraints (construction of the projection matrices, evaluation of the maximum allowable step length, selection of the set of active constraints, etc...).

Therefore it is important to reduce as much as possible the number of behavior constraints retained at each stage of the optimization process. Constraint diletion techniques, such as those proposed by SCHMIT and MIURA [6-8], are simple but effective means of achieving that goal. In this approach only the critical and potentially critical behavior constraint are included in the linearized problem statement at each redesign stage. Potentially critical constraints are defined as those which are close to their allowable limit within a given tolerance (which can be dynamically updated at each stage). Note that the structural reanalysis must be decomposed in two parts : first the constraint values are computed for the real loading conditions, and then the gradients of the retained constraints are evaluated using virtual or pseudo-load cases (see section 3.1). Finally, using zero order approximation for the stress constraints also permits a dramatic reduction in the number of linearized constraints (see section 3.2).

7.3 Program organization

On the basis of the foregoing developments, we propose in this section the program organization that seems the most appropriate for dealing with large structural optimization problems :

(1) Preprocessor

- generation of the element stiffness matrices, mass matrices and implicit loads for unit values of the design variables
- data preparation depending on design variable linking

loop (2) Update of the element stiffness matrices, mass matrices and implicit loads for the current values of the design variables

(3) Static analysis

- triangular decomposition of the system stiffness matrix
- evaluation of the displacements (back substitutions for the real loading cases)
- evaluation of the stresses

(4) Dynamic analysis

- assemblage of the system mass and stiffness matrices
- solution of the eigenproblem, yielding the frequencies and the associated modal displacements

(5) Stability analysis

- generation of the element geometric stiffness matrices (from the stresses previously computed)

- assemblage of the system matrices and solution of the eigenproblem

(6) Scaling of the design variables and convergence checks $\xrightarrow{\text{end}}$ (10)

(7) Linearized problem statement

- selection of the potentially critical constraints
- selection of zero/first order approximation scheme for each stress constraint
- definition of the additional dummy loads (virtual or pseudo load cases)
- evaluation of the constraint gradients (back substitutions for the dummy load cases)

(8) Optimization

- solution of the linearized problem using either primal or dual algorithm

$\xleftarrow{\text{loop}}$(9) Storage of the design variable vector and return to (2)

(10) Print out of the results

End.

In this computer program organization, it is assumed that the search for the optimal design is performed in a fully automatic way. It should be noted, however, that it is often desirable to allow for a human intervention in the redesign process. For example very large structural models, because they are time consuming, cannot be treated in a single run and it is better executing the optimization program stage by stage with intermediate verification of the results. The control parameters can then be reset periodically to adequate values (change of the optimization algorithm, modification of the $\bar{k}$ parameter, specification of new

tolerances for the selection of the linearized constraints, etc...). By storing in a data base all the analysis results at each stage, it is moreover possible to restart the program at any previously generated design point.

In this connection , the concept of automatic redesign should be replaced by that of interactive redesign on a graphic terminal, allowing therefore the designer to easily monitor the optimization process. It is worth pointing out that the high quality explicit approximations of the behavior constraints employed in this work could also be used in the context of computer aided design procedures. Indeed the constraint gradients with respect to the reciprocal design variables can be viewed as "sensibility indices" whose values provide qualitative and even quantitative informations about the way in which the structure should be redesigned.

8. NUMERICAL EXAMPLES

A wide variety of truss and box structures were proposed in Ref. [1] to illustrate the power of the mixed method and generalized optimality criterion concepts applied to stress and displacement constraints. In this section attention is focused on sample problems involving frequency and buckling constraints, as well as problems with beam elements.

Although the results presented in this chapter are only concerned with the SAMCEF program [15], it should be emphasized that other significant problems have been successfully treated with the ACCESS-3 program [13, 14], in which the dual method approach is also available.

8.1. Elastic rod with tip mass

In order to validate the optimization strategy proposed in section 4.3, the following simple problem was considered : minimize the weight of an elastic bar subject to minimum fundamental frequency constraint (longitudinal vibration). The bar is fixed at one end and it supports a given mass at the free end (see Fig. 6).

This example is interesting because it has received an exact analytical solution in the context of continuous models studies [34, 35, 38]. The optimal structure for a prescribed lower limit $\underline{\omega}$ is defined by the cross-sectional area a(x) and the axial displacement u(x) as follows :

$$a(x) = \frac{\beta \bar{m}}{\rho} \text{ th } (\beta \ell) \frac{\text{ch}^2(\beta \ell)}{\text{ch}^2(\beta x)} \qquad (8.1)$$

$$u(x) = \frac{\text{sh}(\beta x)}{\text{sh}(\beta \ell)} \qquad (8.2)$$

with

$$\beta = \underline{\omega} \sqrt{\frac{\rho}{E}} \qquad (8.3)$$

In these expressions, ρ denotes the mass density for the given material, E is its Young modulus, ℓ represents the length of the rod and $\bar{m}$ is the tip mass (at $x = \ell$). The total mass for the optimal structure ($\bar{m}$ non included) is

$$W^{*} = \bar{m}\ \mathrm{sh}^{2}(\beta\ell) \tag{8.4}$$

These analytical results were compared with those generated by using recursively the optimality criterion stated in Eq. (4.36). For a finite element model made up of 10 bars, an optimal weight of 0.6647 kg is produced in 8 iterations. The corresponding analytical solution (8.4) leads to $W^{*} = 0.6646$ kg. The numerical values of the cross-sectional areas and the modal displacements in the finite element solution are almost identical to the exact values provided by Eqs. (8.1) and (8.2) (up to the fourth digit).The iteration history for this 10-bar model is illustrated in Fig. 7.

The same problem was also treated with a cruder finite element model involving only 4 bars. The optimum is attained in 9 iterations. Table I shows the detailed iteration history. The final design weighs 0.6656 kg and it remains close to the analytical solution despite the simplicity of the discretization (only 4 degrees of freedom). The variations of the cross-sectional area and of the axial displacement are depicted in Fig. 6. The agreement between the numerical results provided by the optimality criterion (4.36) and the analytical solution given by Eqs. (8.1, 8.2) is excellent.

8.2. Sandwich beam with Euler buckling contraint

Attention is now directed to the Euler sandwich column shown in Fig. 8. The beam has length ℓ and is hinged at both ends. The problem consists in minimizing its weight for a given critical buckling load P (see Fig. 8). For a sandwich beam with rectangular cross-section, specific weight and stiffness properties (per unit length) are linearly related, so that the problem can be treated by using the same strategies as in the case of thin-walled structures.

Considering thus a rectangular sandwich beam with constant core height 2h and constant width b and assuming identical upper and lower face sheets with variable thickness $t \ll h$, it follows that the specific bending stiffness can be written

$$s = 2 E b h^2 t \tag{8.5}$$

and that the flange specific weight takes the form

$$w = 2 \rho b t \tag{8.6}$$

where E and ρ represent respectively the Young's modulus and the mass density for the flange material. Note that only the contribution of the face sheets to the structural weight can be minimized, because the core sizes are kept constant. The onjective function is therefore linear in the design variable t (sheet thickness).

The analytical solution to this problem was obtained by PRAGER and TAYLOR [38]. At the optimum the specific stiffness s(x) and the lateral displacement u(x) that characterizes the critical buckling mode are expressed as follows :

$$s(x) = \frac{P}{2} x (\ell - x) \tag{8.7}$$

$$u(x) = \frac{4}{\ell^2} x (\ell - x) \tag{8.8}$$

The sheet thickness can then be evaluated using Eq. (8.5) and Eq. (8.7). Integrating the specific weight stated in Eq. (8.6) along the length of the column yields the optimal weight

$$W^* = \frac{P \rho \ell^3}{12 E h^2} \tag{8.9}$$

If W° denotes the weight of a beam with uniform thickness, supporting the same buckling load P, it follows that

$$W^* = \frac{\pi^2}{12} W^\circ \approx 0.822 W^\circ$$

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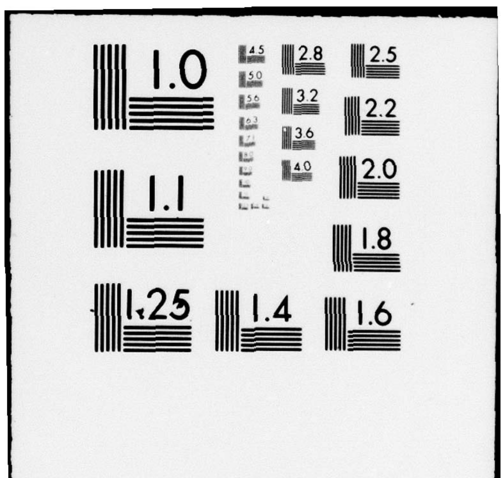

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The latter expression demonstrates the benefit gained from optimizing the column.

Using a finite element model involving 10 equal length segments, the discretized solution resulting from applying the optimality criterion redesign relations (see Eq. 5.29) is generated in 6 iterations. Table 2 reproduces the iteration history. Note that by symmetry only 5 beam elements are necessary to describe the problem. The final design weighs 265.7 kg, while the analytical solution (8.9) leads to 260 kg. The continuous and discretized solutions are compared in Fig. 8 (see also Table 2). It can be seen that the numerical results are very close to the analytical solution, although the finite element model involves only 10 degrees of freedom.

8.3. Euler column with rectangular cross-section

It has been shown in section 6.1.1 that the optimum design of beam structures depends upon the relation between the moment of inertia and the cross-sectional area of the beam members. In order to illustrate the various possibilities permitted by Eq. (6.1), we consider again the Euler column problem, but we assume now a rectangular cross-section with height h and width b. As explained in section 6.1.1, the case where b is variable and h is constant reduces to the previously examined problem, because the bending stiffness for each member is linear with respect to the cross-sectional area. Now, if b is kept constant and h is variable, the relation (6.5) holds, in which case one should choose $p = 3$ in the expression (6.8) of the stiffness matrix. Finally if the cross-section shape remains constant along the beam axis (i.e. fixed h/b ratio) one should adopt $p = 2$ as shown by Eq. (6.6).

From the following remarks it follows that the optimum design must be sought by applying the optimality criterion redesign relations (6.27) with $p = 1$, 2 or 3 depending upon the hypothesis made on the variable quantity. Of course it should be kept in mind that u represents here the critical buckling load instead of a flexibility. Consequently the coefficients c_{ij} must be corrected according

to Eq. (5.25), in order to take the scaling effect into consideration. Since only one behavior constraint is imposed in the Euler column problem statement, the redesign relation (6.35) is directly applicable once the adequate value of p is selected.

In each of the three cases considered the finite element model involves 5 beam members (symmetry). In the initial design the width and the height of the rectangular cross-section are taken as 10 and 1, respectively, which leads to a weight equal to 10 (arbitrary unit system ; see Fig. 9). After scaling of the design variables, the weight of the strictly critical, feasible design takes on different values in the three cases considered, because the scaling factor multiplies the moments of inertia of each beam element, rather than their cross-sectional areas.

Table 3.a contains the iteration histories for each three situations $p = 1$, $p = 2$ and $p = 3$. Only four to six structural reanalyses are required to achieve convergence. Table 3.b represents the corresponding final designs, which all satisfy the optimality criterion stated in Eq. (6.30). The least weight design is obtained when the width of the cross-section is varied ($p = 1$). From the tip to the middle of the beam, the moment of inertia (and thus the cross-sectional area) is increased by a factor of five. The heaviest design is obtained by choosing the height of the cross-section as the variable quantity ($p = 3$). Now the moment of inertia is modified by a factor of ten along the beam. However, the change in cross-sectional area is only by a factor of two, which explains the higher value of the optimal weight in this case. The variations of the moment of inertia and of the cross-sectional area are displayed in Fig. 9 for the three cases considered.

8.4. I-beam with multiple frequency constraints

The sample problems examined in the previous sections are concerned with a single behavior constraint, in which case each explicit problem generated in sequence can be solved in closed form (see for example section 6.3). When multiple behavior constraints are simultaneously active, application of the generalized optimality

criterion stated in Eqs. (2.21, 2.25) requires the explicit problems to be solved by using dual algorithms. As an example we consider the I-beam represented in Fig. 10. Design data are given in Table 4, including the lower and upper limits imposed on the natural frequencies.

The problem consists in minimizing the weight of the beam while controlling the frequencies of its three first eigenmodes :

. flange flexion (along the Y-axis)
. torsion (with respect to the X-axis)
. web flexion (along the Z-axis)

From the structural analysis point of view, a difficulty in this problem is to find a model made up of membrane elements, that is capable of representing properly each three vibration modes. The two bending modes are well taken into account in a membrane model. However the torsional mode is not correctly represented, because it reduces to differential flexion of the two flanges (with the same frequency as the fundamental mode). It is therefore necessary to include fictitious diaphragms in the analysis model, so as to prevent differential flange flexion. These diaphragms are introduced with zero mass density.

Table 5 reproduces the natural frequencies of the three first eigenmodes, evaluated on one hand,with pure membrane models (with and without diaphragms) and, on the other hand, with a much more accurate model made up of flat shell elements. In both cases, the beam is subdivided into five equal length segments, which leads to 25 rectangular elements (plus 10 diaphragms in the third model considered in Table 5). The model retained yields excellent results, which are in good agreement with those generated using flat shell elements. It involves 250 degrees of freedom, because a second degree displacement field is taken in each membrane element. In the numerical solution of the eigenproblem all the interface degrees of freedom are condensed, so that only 87 degrees of freedom are retained. It should be noted that in the optimization process the

thickness of the diaphragms are fixed to their initial value (1 mm), so that the problem involves 25 design variables.

8.4.1. Case A : inequality constraints on the natural frequencies

In a first optimization exercise, lower and upper bounds are imposed on each three eigenfrequencies (see Table 3). Table 6 and Fig. 11 summarize the iteration histories by reproducing the variations of the weight and frequencies with the number of structural reanalyses. The convergence of the optimization process is remarkably rapid. Only 5 analyses are sufficient to generate an optimum design, weighing 393.7 kg. The fundamental frequency does not reach the prescribed lower bound (1.0 Hz) nor the upper bound (1.2 Hz). The two higher frequencies are equal to their respective minimal allowable value (1.2 Hz and 2.5 Hz).

The final design is given in Table 7. Note that the thickness in the upper flange differs from those in the lower flange, because of the support condition at the fifth of the beam length. Also an increase in the flanges thicknesses appears at the support level. With regard to the web thickness it can be seen that it is lower than the flange thickness, mainly at the hinged end neighborhood.

8.4.2. Case B : equality constraints on the natural frequencies

In a second optimization exercise, equality constraints were assigned to each three frequencies, in order to reduce the fundamental frequency to 1.0 Hz, while keeping the two other frequencies at 1.2 Hz and 2.5 Hz, respectively. The iteration history presented in Table 6 and illustrated in Fig. 11 again demonstrates the remarkable efficiency of the generalized optimality criterion. The increase in weight after the first r design stage is due to the equality constraint which is now imposed on the first frequency. Note that only 6 structural reanalyses are required to achieve convergence and that the equality constraints are satisfied within less than 10^{-4} relative accuracy.

Table 7 represents the final designs. Although the weights btained in the two cases considered are very close to each other (393.7 kg in case A and 394.1 kg in case B), the material distribution is quite different in case B, because of the decrease in the fundamental frequency. The differences in thickness between the two flanges are more pronounced. Moreover the web thickness varies more strongly along the beam axis.

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TABLE 1

ITERATION HISTORY FOR ELASTIC ROD

iteration	weight (kg)	frequency (Hz)	cross-sectional area (cm^2)				modal displacement		
			a_1	a_2	a_3	a_4	u_2	u_3	u_4
1	0.7800	631.5	1.0	1.0	1.0	1.0	0.2757	0.5409	0.7854
2	0.6469	583.9	1.0605	0.9771	0.8004	0.4793	0.1946	0.3993	0.6346
3	0.6599	596.9	0.9513	0.8927	0.7996	0.7407	0.2404	0.4881	0.7470
4	0.6644	599.3	1.0073	0.9414	0.8215	0.6371	0.2246	0.4569	0.7062
5	0.6653	599.8	0.9981	0.9230	0.8096	0.6908	0.2316	0.4713	0.7272
6	0.6655	599.9	0.9980	0.9321	0.8165	0.6663	0.2285	0.4649	0.7177
7	0.6656	600.0	0.9937	0.9281	0.8132	0.6782	0.2299	0.4679	0.7223
8	0.6656	600.0	0.9958	0.9300	0.8148	0.6726	0.2292	0.4665	0.7202
9	0.6656	600.0	0.9948	0.9291	0.8140	0.6752	0.2296	0.4672	0.7212
analytical solution	0.6646	600.0	0.9952	0.9295	0.8144	0.6747	0.2296	0.4672	0.7211

TABLE 2

ITERATION HISTORY FOR SANDWICH BEAM

iteration	weight (kg)	critical load	face sheets thickness (cm)					modal displacement			
			a_1	a_2	a_3	a_4	a_5	u_2	u_3	u_4	u_5
1	390.0	1.2337	1.0000	1.0000	1.0000	1.0000	1.0000	0.3090	0.5878	0.8090	0.9511
2	261.9	0.9826	0.1874	0.4792	0.7378	0.9268	1.0264	0.3722	0.6520	0.8469	0.9620
3	265.6	0.9993	0.2212	0.5257	0.7586	0.9118	0.9877	0.3591	0.6390	0.8393	0.9598
4	265.7	0.9999	0.2151	0.5181	0.7571	0.9179	0.9987	0.3615	0.6414	0.8408	0.9602
5	265.7	1.0000	0.2162	0.5196	0.7575	0.9169	0.9967	0.3610	0.6410	0.8405	0.9601
6	265.7	1.0000	0.2160	0.5193	0.7574	0.9171	0.9971	0.3611	0.6410	0.8406	0.9601
analytical solution	260.0	1.0000	0.1900	0.5100	0.7500	0.9100	0.9900	0.3600	0.6400	0.8400	0.9600

TABLE 3

EULER COLUMN WITH RECTANGULAR CROSS-SECTION

a. Iteration history

iteration	h constant		h/b constant		b constant	
	weight	scaling factor	weight	scaling factor	weight	scaling factor
1	9.7267	0.9727	9.8624	0.9727	9.9080	0.9727
2	8.2008	1.0177	8.7805	1.0439	9.0838	1.0662
3	8.1775	1.0007	8.7376	1.0046	9.0343	1.0104
4	8.1766	1.0000	8.7335	1.0004	9.0274	1.0012
5			8.7332	1.0000	9.0266	1.0002
6			8.7331	1.0000	9.0264	1.0000

b. Final designs

element	I + a (h constant)				I + a^2 (h/b constant)				I + a^3 (b constant)			
	I	a	b	h	I	a	b	h	I	a	b	h
1	0.2160	2.592	2.592	1.0	0.1550	4.313	0.6567	6.567	0.1343	5.442	10.0	0.5442
2	0.5194	6.233	6.233	1.0	0.4785	7.578	0.8705	8.705	0.4651	8.233	10.0	0.8233
3	0.7574	9.089	9.089	1.0	0.7673	9.596	0.9796	9.796	0.7770	9.769	10.0	0.9769
4	9.9171	11.005	11.005	1.0	0.9727	10.804	1.0394	10.394	1.0044	10.642	10.0	1.0642
5	0.9970	11.964	11.964	1.0	1.0784	11.376	1.0666	10.666	1.1229	11.045	10.0	1.1045

I : moment of inertia b : width

a : cross-sectional area h : height

TABLE 4

DESIGN DATA FOR I-BEAM PROBLEM

Material	:	steel
Young's modulus	:	$E = 2.10^5$ N/mm^2
Poisson's ratio	:	$\nu = 0.3$
mass density	:	$\rho = 7.8\ 10^{-6}$ kg/mm^3
minimum thickness	:	$\underline{a} = 1$ mm
initial thickness	:	$a^o = 10$ mm
non-structural mass	:	20.000 kg (see Fig. 10)

case	mode	frequency limits (Hz)		type of constraint
		minimal	maximal	
Case A	1	1.0	1.2	inequality
	2	1.2	2.5	
	3	2.5	/	
Case B	1	1.0	1.0	equality
	2	1.2	1.2	
	3	2.5	2.5	

TABLE 5

COMPARISON OF VARIOUS ANALYSIS MODELS FOR I-BEAM

mode	membrane 1st degree	membrane 2nd degree		flat shell 2nd degree
		without diaphragm	with diaphragm	
1 flexion (Y)	1.9782	0.9285	0.9285	0.9286
2 torsion (X)	1.9782	0.9285	1.0512	1.0379
3 flexion (Z)	2.9515	2.4970	2.4970	2.4951

TABLE 6

ITERATION HISTORY FOR I-BEAM PROBLEM

Iteration	case A : inequality constraints				case B : equality constraints			
	weight (kg)	frequencies (Hz)			weight (kg)	frequencies (Hz)		
		1	2	3		1	2	3
1	507.00	0.9285	1.051	2.497	507.00	0.9285	1.051	2.497
2	443.14	1.137	1.241	2.614	565.56	1.003	1.253	2.679
3	397.99	1.102	1.202	2.514	432.78	0.9849	1.250	2.531
4	394.11	1.101	1.200	2.501	397.17	1.003	1.205	2.508
5	393.74	1.103	1.200	2.500	394.06	1.000	1.200	2.501
6					393.69	1.000	1.200	2.500

TABLE 7

FINAL DESIGNS FOR I-BEAM PROBLEM

thickness (mm) : case A (inequality constraints) / case B (equality constraints)

upper flange

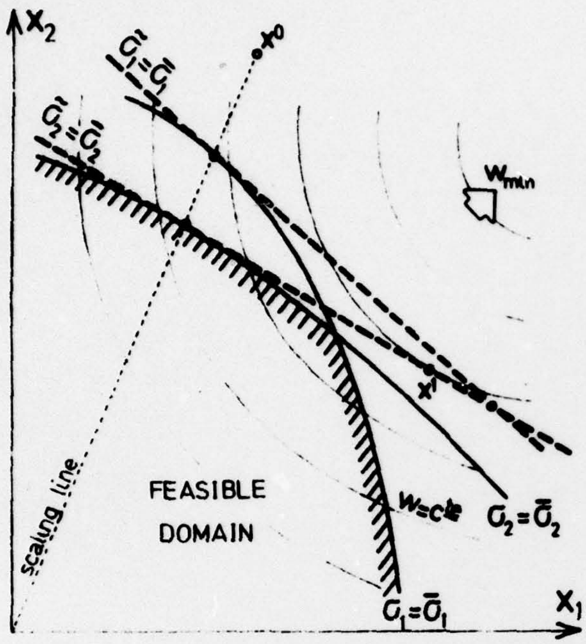
15.795 10.237	17.298 14.703	11.671 9.796	6.143 4.938	1.815 1.320

web

hinged						free
	5.101 9.064	3.602 4.259	3.329 3.869	3.294 3.835	1.997 2.321	

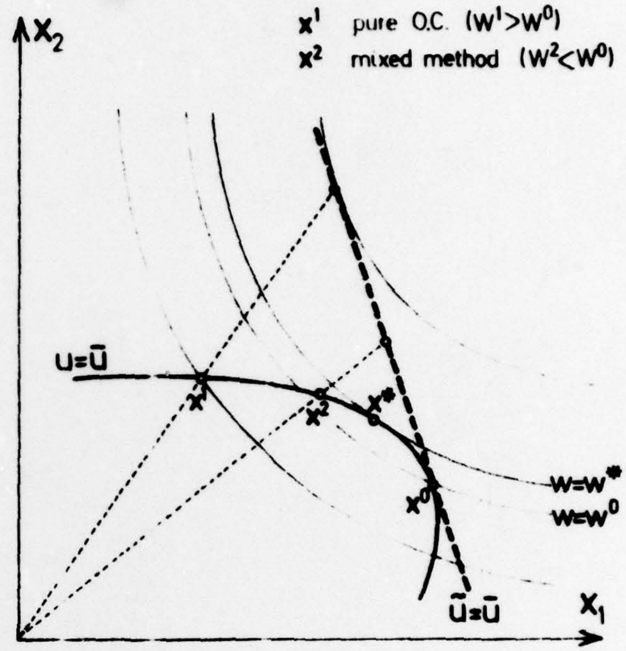
lower flange

15.403 16.135	17.009 18.623	11.545 12.608	6.074 6.640	1.792 1.963



THE LINEARIZED PROBLEM

FIG. 1



CONVERGENCE OF THE MIXED METHOD

FIG. 2

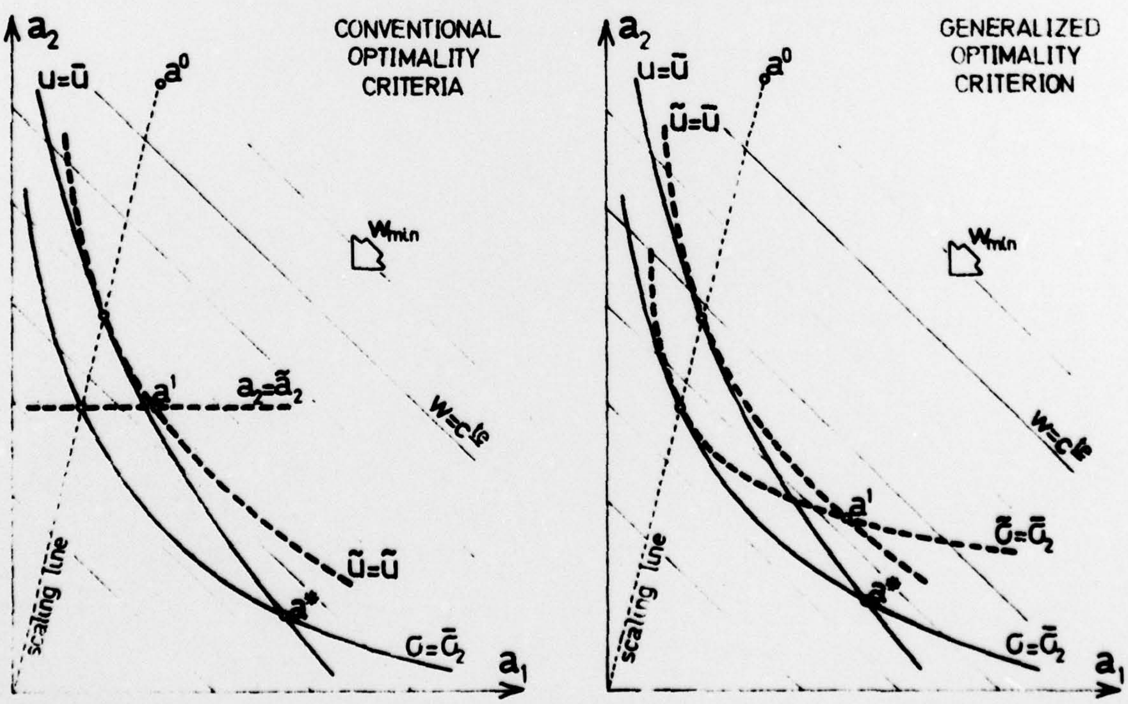


FIG. 3 COMPARISON OF FIRST AND ZERO ORDER APPROXIMATIONS

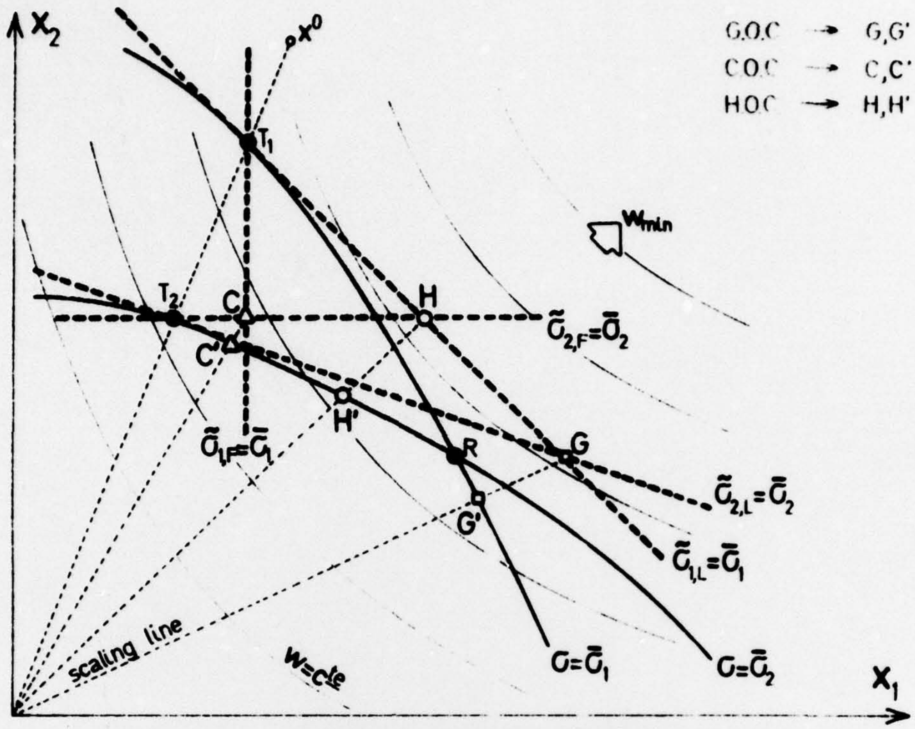


FIG. 4 GENERALIZED, CONVENTIONAL AND HYBRID OPTIMALITY CRITERIA

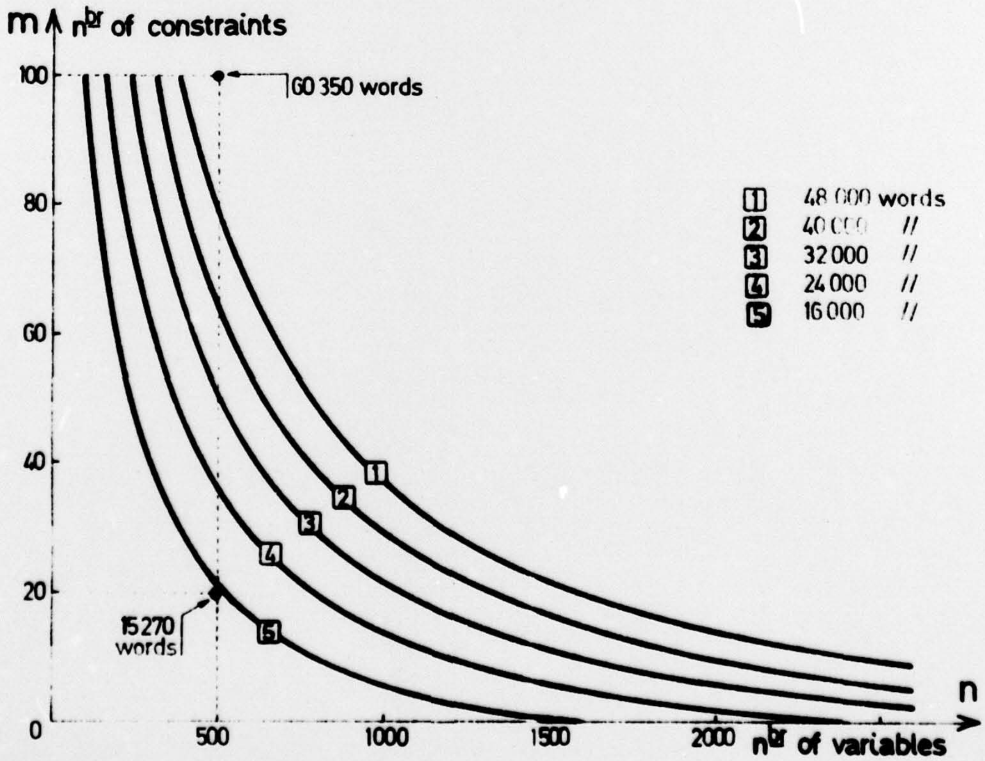


FIG. 5 CORE REQUIREMENT FOR OPTIMIZATION ALGORITHMS

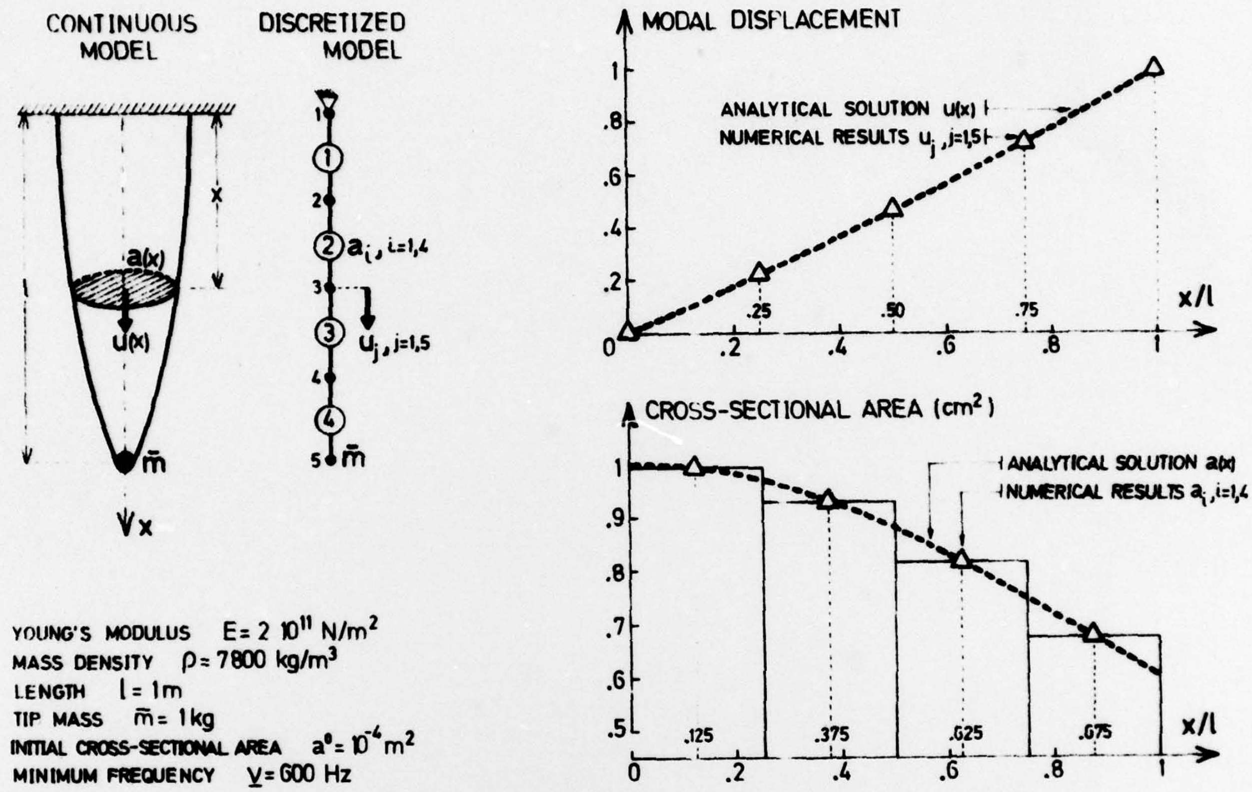


FIG. 6 ELASTIC ROD WITH TIP MASS

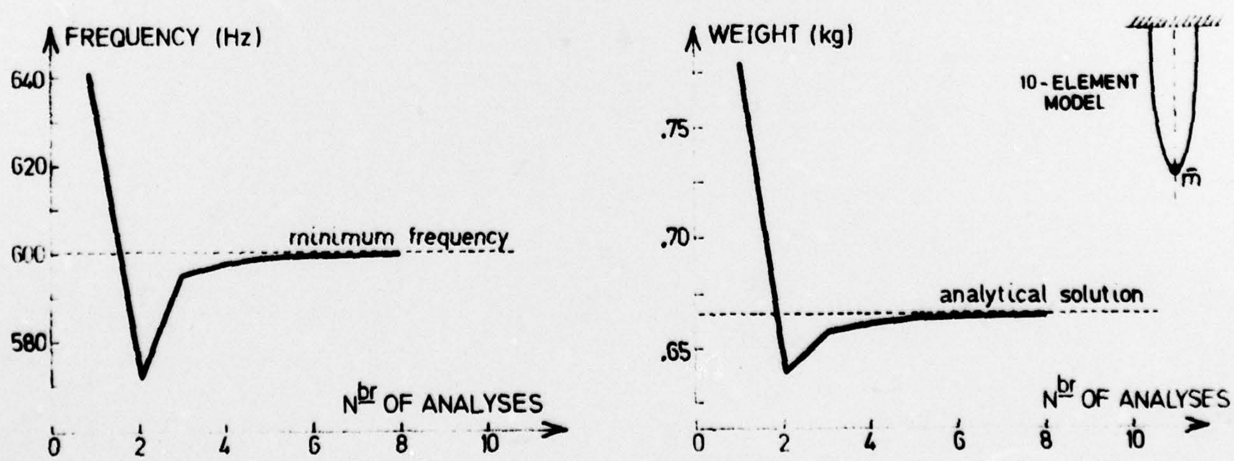
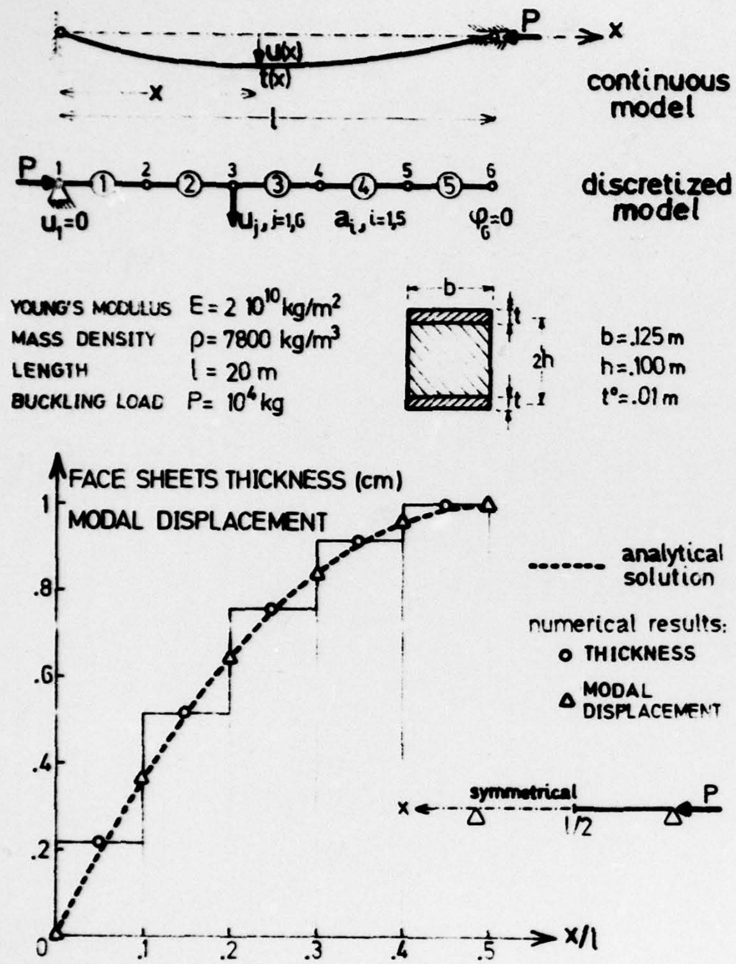
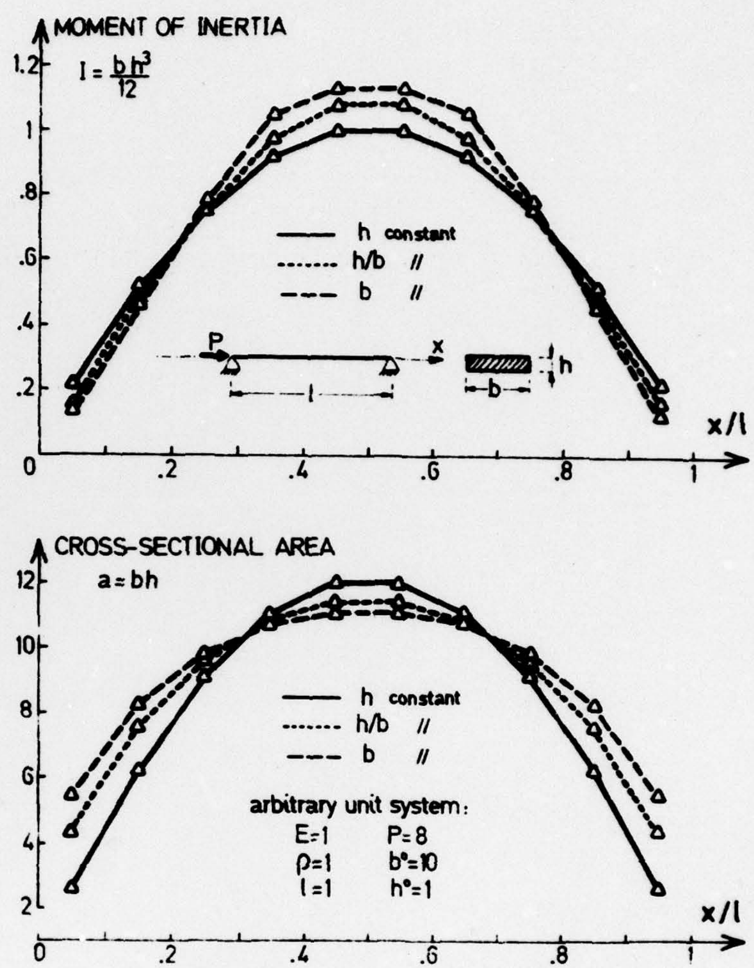


FIG. 7 ITERATION HISTORY FOR ELASTIC ROD



SANDWICH BEAM WITH EULER BUCKLING CONSTRAINT

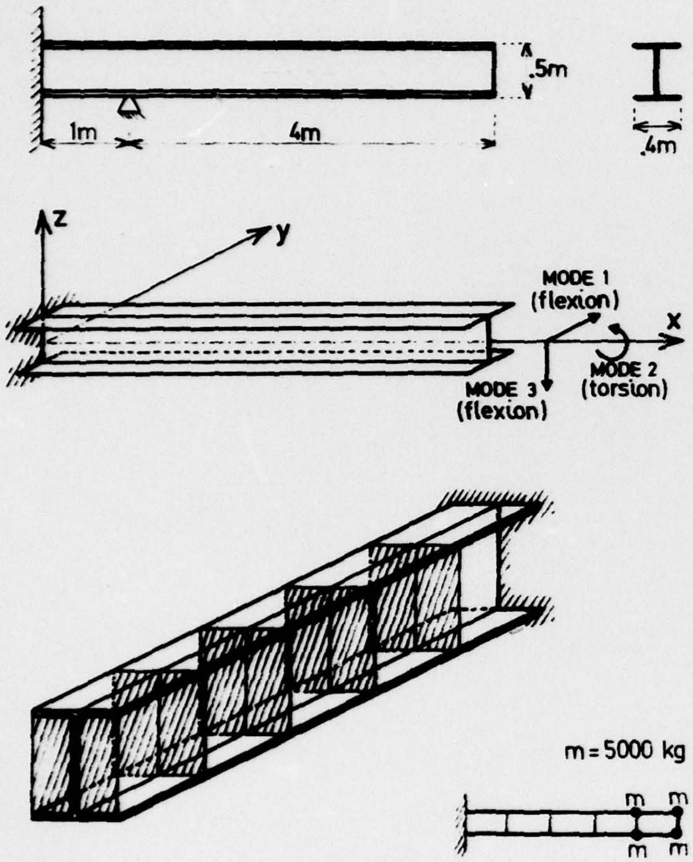
FIG. 8



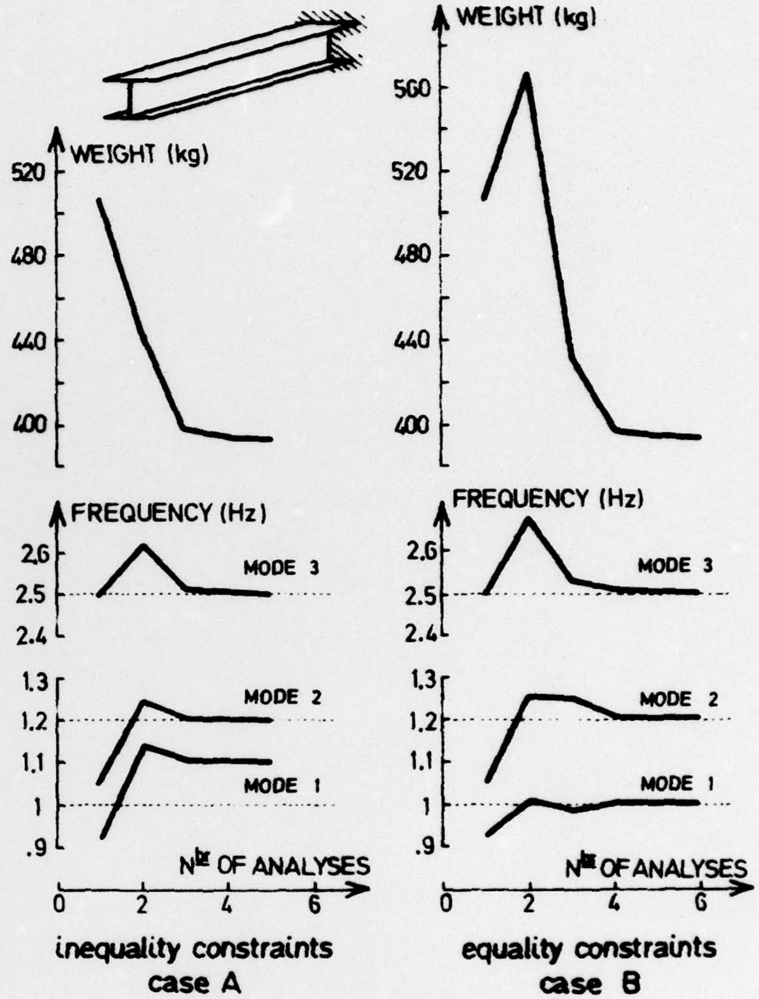
EULER COLUMN WITH RECTANGULAR CROSS-SECTION

FIG. 9

25 membrane elements
10 fictitious elements (diaphragms without mass)
15 design variables
250 degrees of freedom



I-BEAM MODEL
FIG. 10



ITERATION HISTORY FOR I-BEAM
FIG. 11